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Explicit formula for the Γ-convergence homogenised quadratic curvature energy in isotropic Cosserat shell models

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Abstract. We show how to explicitly compute the homogenised curvature energy appearing in the isotropic Γ-limit for flat and for curved initial configuration Cosserat shell models, when a parental three-dimensional minimisation problem on $\Omega \subset \mathbb{R}^3$ for a Cosserat energy based on the second-order dislocation density tensor $\alpha := \overline{R}^T \operatorname{Curl}\overline{R} \in \mathbb{R}^{3\times 3}$, $\overline{R} \in \mathrm{SO}(3)$ is used.

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Contents

1. Introduction

The Cosserat theory introduced by the Cosserat brothers in 1909 [13,15] represents a generalisation of the elasticity theory. While the elasticity theory models each constituent particle of the body as a material point, i.e. it is able to model only the translation of each particle through the classical deformation $\varphi\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, the Cosserat theory models the micro-rotation of each particle attaching to each material point an independent triad of orthogonal directors, the microrotation $\overline{R}\colon \Omega \subset \mathbb{R}^3 \to \mathrm{SO}(3)$. Invariance of the energy under superposed rigid body motions (left invariance under SO(3)) allowed them to conclude the suitable form of the energy density as $W = W(\overline{U}, \mathfrak{K})$, where $\overline{U} := \overline{R}^T \mathrm{D}\varphi$ is the first Cosserat deformation tensor and $\mathfrak{K} := (\overline{R}^T \partial_{x_1}\overline{R}, \overline{R}^T \partial_{x_2}\overline{R}, \overline{R}^T \partial_{x_3}\overline{R})$ is the second Cosserat deformation tensor. The Cosserat brothers never considered specific forms of the elastic energy, and they never linearised their model to obtain the well-known linear Cosserat (micropolar) model [26]. In the present paper, we only consider isotropic material, i.e. the behaviour of the elastic material is modelled with the help of an additionally right-invariant energy under SO(3). In addition, we will consider quadratic energies in suitable strains (a physically linear dependence of the stress tensor and of the couple stress tensor on the strain measures) which allows an explicit and practical [30,41] representation of the energy.

In [40], we have provided a nonlinear membrane-like Cosserat shell model on a curved reference configuration starting from a geometrically nonlinear, physically linear three-dimensional isotropic Cosserat model. Beside the change of metric, the obtained membrane-like Cosserat shell model [40] is still capable to capture the transverse shear deformation and the Cosserat curvature due to remaining Cosserat effects. The Cosserat shell model presented in [40] for curved initial configuration generalises the Cosserat shell model constructed in [33] for flat initial configurations. There are many different ways to mathematically model shells [28], e.g. the *derivation approach* [20–24,31,32], the *intrinsic approach* [1,2,27], the *asymptotic method*, the *direct approach*[3–6,8–11,15,18,25,27,39]). However, Γ-*convergence* methods are preferred in the mathematical community.

When the Cosserat parental three-dimensional energy is considered, in the deduction of the Gamma-limit for the curved initial configuration we have to construct four homogenised energies, while in the expression of the Gamma-limit will appear only two: the homogenised membrane energy and the homogenised curvature energy. In the deduction of the Gamma-limit in [40], we have explicitly stated the form of the homogenised membrane energy, while the explicit form of the homogenised curvature energy was only announced and we have used only some implicit properties (its continuity, convexity, etc.). The same was done in the deduction of the Gamma-limit for a flat initial configuration in [33] (we notice that another form of the Cosserat curvature energy was considered) and no explicit form of the homogenised curvature energy could be given. In [40], we have announced the form of the homogenised curvature energy without giving details about its deduction. Therefore, this is the main aim of this paper, i.e. to provide the solutions of all optimisation problems needed for having an explicit approach of Cosserat shell model for flat (Sect. 3) and curved (Sect. 4) initial configurations via the Gamma-limit. The second goal is to point out the advantages, at least from a computation point of view, of the usage of the curvature strain tensor $\alpha := \overline{R}^T \mathrm{Curl}\overline{R}$ in the parental three-dimensional Cosserat curvature energy, instead of other curvature strain tensors considered in literature. We mention that, even if α is controlled by $\widehat{\mathfrak{K}} := \Big(\overline{R}^T \mathrm{D}(\overline{R}.e_1), R^T \mathrm{D}(\overline{R}.e_2), R^T \mathrm{D}(\overline{R}.e_3)\Big) \in \mathbb{R}^{3\times 9}$ used in [31,33], an explicit expression via Γ-convergence of the homogenisation of the quadratic curvature energy in terms of the third-order tensor $\widehat{\mathfrak{K}}$ is missing in the literature, even for flat initial configuration. In fact, it turned out that $\widehat{\mathfrak{K}}$ is frame indifferent but not itself isotropic, a fact which makes it unsuitable to be used in an isotropic model. Beside these advantages of the usage of the second-order dislocation density tensor α, there is a one-two-one relation between α and the so-called wryness tensor $\Gamma := \Big(\mathrm{axl}(\overline{R}^T\, \partial_{x_1}\overline{R}) \,|\, \mathrm{axl}(\overline{R}^T\, \partial_{x_2}\overline{R}) \,|\, \mathrm{axl}(\overline{R}^T\, \partial_{x_3}\overline{R})\Big) \in \mathbb{R}^{3\times 3}$ (second-order Cosserat deformation tensor [15], a Lagrangian strain measure for curvature orientation change[16]). This property is not shared with $\widehat{\mathfrak{K}}$ and $\mathfrak{K}$. We show that considering $\mathfrak{K}$ is equivalent to a particular choice of the constitutive coefficients in a model in which α is used and, therefore, the formulas determined for the homogenised quadratic curvature energy via Gamma convergence are valid for parental isotropic three-dimensional energies which are quadratic in $\mathfrak{K}$, too. However, the general form of a quadratic isotropic energy has a more complicated form in terms of $\mathfrak{K}$ in comparison with the case when we express it in terms of α, see Sect. 2.3. Therefore, from a computational point of view it is more convenient to consider α in a isotropic Cosserat model. Moreover, using [35] (see Sect. 2.3), we have that α controls $\mathfrak{K}$ in $L^2(\Omega, \mathbb{R}^{3\times3\times3})$ which controls $\widehat{\mathfrak{K}}$ in $L^2(\Omega, \mathbb{R}^{3\times3\times3})$ which controls α in $L^2(\Omega, \mathbb{R}^{3\times3})$. Therefore, a positive definite quadratic form in terms of one of the three variants of appropriate curvature tensors is energetically controlled by each of the three Cosserat curvature tensors. This is why when the problem of existence of the solution or other similar qualitative results are considered, the form of the used Cosserat curvature strain tensor is irrelevant, in the sense that if such a result is obtained for a

Cosserat curvature quadratic in Cosserat curvature strain tensor, then it may be immediately extended for Cosserat curvature quadratic energies in the other two Cosserat curvature strain tensor considered in the present paper. However, the usage of the Cosserat curvature strain tensor α has two main advantages:

- A quadratic isotropic energy in terms of the wryness tensor Γ is rewritten in a transparent and explicit form as a quadratic energy in terms of dislocation density tensor α (and vice versa), see Sect. 2.3;
- The expression of a quadratic isotropic energy in terms of the wryness tensor Γ is very simple and suitable for analytical computations, see Sect. 2.3;
- It admits the explicit analytical calculation of the homogenised quadratic curvature energies in the construction of the Cosserat shell model via Γ-convergence method, see Sects. 3 and 4.

2. Three-dimensional geometrical nonlinear and physical linear Cosserat models

2.1. General notation

Before continuing, let us introduce the notation we will use or have already used in Sect. 1 and in abstract. We denote by $\mathbb{R}^{m\times n}$, $n,m\in\mathbb{N}$, the set of real $m\times n$ second-order tensors, written with capital letters. We adopt the usual abbreviations of Lie-group theory, i.e. $\mathrm{GL}(n)=\{X\in\mathbb{R}^{n\times n}\,|\,\det(X)\neq 0\}$ the general linear group, $\mathrm{SL}(n)=\{X\in \mathrm{GL}(n)\,|\,\det(X)=1\}$, $\mathrm{O}(n)=\{X\in \mathrm{GL}(n)\,|\,X^TX=\mathbb{1}_n\}$, $\mathrm{SO}(n)=\{X\in \mathrm{GL}(n)|X^TX=\mathbb{1}_n,\det(X)=1\}$ with corresponding Lie-algebras $\mathfrak{so}(n)=\{X\in\mathbb{R}^{n\times n}\,|X^T=-X\}$ of skew-symmetric tensors and $\mathfrak{sl}(n)=\{X\in\mathbb{R}^{n\times n}\,|\mathrm{tr}(X)=0\}$ of traceless tensors. Here, for $a,b\in\mathbb{R}^n$ we let $\big\langle a,b\big\rangle_{\mathbb{R}^n}$ denote the scalar product on $\mathbb{R}^n$ with associated

(squared)

vector norm $\|a\|^2_{\mathbb{R}^n}=\big\langle a,a\big\rangle_{\mathbb{R}^n}$. The standard Euclidean scalar product on $\mathbb{R}^{n\times n}$ is given by $\big\langle X,Y\big\rangle_{\mathbb{R}^{n\times n}}=\mathrm{tr}(XY^T)$, and thus, the

(squared)

Frobenius tensor norm is $\|X\|^2=\big\langle X,X\big\rangle_{\mathbb{R}^{n\times n}}$. In the following, we omit the index $\mathbb{R}^n,\mathbb{R}^{n\times n}$. The identity tensor on $\mathbb{R}^{n\times n}$ will be denoted by $\mathbb{1}_n$, so that $\mathrm{tr}(X)=\big\langle X,\mathbb{1}_n\big\rangle$. We let $\mathrm{Sym}(n)$ and $\mathrm{Sym}^+(n)$ denote the symmetric and positive definite symmetric tensors, respectively. For all $X\in\mathbb{R}^{3\times 3}$ we set $\mathrm{sym}\,X=\frac{1}{2}(X^T+X)\in\mathrm{Sym}(3)$, $\mathrm{skew}\,X=\frac{1}{2}(X-X^T)\in\mathfrak{so}(3)$ and the deviatoric part $\mathrm{dev}\,X=X-\frac{1}{n}\,\mathrm{tr}(X)\,\mathbb{1}_n\in\mathfrak{sl}(n)$ and we have the orthogonal Cartan decomposition of the Lie-algebra $\mathfrak{gl}(3)=\{\mathfrak{sl}(3)\cap\mathrm{Sym}(3)\}\oplus\mathfrak{so}(3)\oplus\mathbb{R}\!\cdot\!\mathbb{1}_3,\ X=\mathrm{dev}\,\mathrm{sym}\,X+\mathrm{skew}\,X+\frac{1}{3}\mathrm{tr}(X)\,\mathbb{1}_3$ .We use the canonical identification of $\mathbb{R}^3$ with $\mathfrak{so}(3)$, and, for$A=\begin{pmatrix}0&-a_3&a_2\\ a_3&0&-a_1\\ -a_2&a_1&0\end{pmatrix}\in\mathfrak{so}(3)$ we consider the operators $\mathrm{axl}:\mathfrak{so}(3)\to\mathbb{R}^3$ and $\mathrm{anti}:\mathbb{R}^3\to\mathfrak{so}(3)$ through $\mathrm{axl}\,A:=(a_1,a_2,a_3)^T$, $A.\,v=(\mathrm{axl}\,A)\times v$, $(\mathrm{anti}(v))_{ij}=-\epsilon_{ijk}\,v_k\ \ \forall\,v\in\mathbb{R}^3$, $(\mathrm{axl}\,A)_k=-\frac{1}{2}\,\epsilon_{ijk}A_{ij}=\frac{1}{2}\,\epsilon_{kij}A_{ji}$, $A_{ij}=-\epsilon_{ijk}\,(\mathrm{axl}\,\mathbf{A})_k=:\mathrm{anti}(\mathrm{axl}\,A)_{ij}$, where ϵ_{ijk} is the totally antisymmetric third-order permutation tensor. For $X\in\mathrm{GL}(n)$, $\mathrm{Adj}(X)$ denotes the tensor of transposed cofactors, while the (i,j) entry of the cofactor is the (i,j)-minor times a sign factor. Here, given $z_1,z_2,z_3\in\mathbb{R}^{n\times k}$, the notation $(z_1\,|\,z_2\,|\,z_3)$ means a matrix $Z\in\mathbb{R}^{n\times 3k}$ obtained by taking z_1,z_2,z_3 as block matrices. A third-order tensor $A=(A_{ijk})\in\mathbb{R}^{3\times3\times3}$ will be replaced with an equivalent object, by reordering its components in a $\mathbb{R}^{3\times 9}$ matrix $A\equiv(A_1\,|\,A_2\,|\,A_3)\in\mathbb{R}^{3\times 9},\ A_k:=(A_{ijk})_{ij}=A.\,e_k\in\mathbb{R}^{3\times 3},\ k=1,2,3$, and we consider $\mathrm{sym}A=\big(\mathrm{sym}\,A_1\,|\,\mathrm{sym}\,A_2,\,|\,\mathrm{sym}\,A_3\big)\in\mathbb{R}^{3\times 9}$, $\mathrm{skew}A=\big(\mathrm{skew}A_1\,|\,\mathrm{skew}A_2\,|\,\mathrm{skew}A_3\big)\in\mathbb{R}^{3\times 9}$, $\mathrm{tr}(A)=\mathrm{tr}(A_1)+\mathrm{tr}(A_2)+\mathrm{tr}(A_3)$. Moreover, we define the products of a second-order tensor $B=(B_{ij})_{ij}\in\mathbb{R}^{3\times 3}$ and a third-order tensor $A=(A_1\,|\,A_2\,|\,A_3)\in\mathbb{R}^{3\times 9}$ in a natural way as $B\,A=(B\,A_1\,|\,B\,A_2\,|\,B\,A_3)\in\mathbb{R}^{3\times 9}$, $A\,B=\big(\sum_{k=1}^3A_k\,B_{k1}\,|\,\sum_{k=1}^3A_k\,B_{k2}\,|\,\sum_{k=1}^3A_k\,B_{k3}\big)\in\mathbb{R}^{3\times 9}$. Let us remark that for $B=(B_{ij})_{ij}\in\mathrm{GL}^+(3)$

having the inverse $B=(B^{ij})_{ij}$ and $A,C\in\mathbb{R}^{3\times 9}$ the following equivalences hold true $A\,B=C \ \Leftrightarrow \ \sum_{k=1}^{3}A_k\,B_{kl}=C_l \ \Leftrightarrow \ \sum_{l=1}^{3}(B^{lm}\sum_{k=1}^{3}A_k\,B_{kl})=\sum_{l=1}^{3}C_l\,B^{lm} \ \Leftrightarrow \ A=C\,B^{-1}$. We define the norm of a third-order tensor $A=(A_1\,|\,A_2\,|\,A_3)\in\mathbb{R}^{3\times 9}$ by $\|A\|^2=\sum_{k=1}^{3}\|A_k\|^2$. For $A_1,A_2,A_3\in\mathfrak{so}(3)$, we define $\operatorname{axl}A=(\operatorname{axl}A_1\,|\,\operatorname{axl}A_2\,|\,\operatorname{axl}A_3)\in\mathbb{R}^{3\times 3}$, while for $z=(z_1\,|\,z_2\,|\,z_3)\in\mathbb{R}^{3\times 3}$ we define $\operatorname{anti}z=(\operatorname{anti}z_1\,|\,\operatorname{anti}z_2\,|\,\operatorname{anti}z_3)\in\mathbb{R}^{3\times 9}$. For a given matrix $M\in\mathbb{R}^{2\times 2}$, we define the lifted quantity $M^\flat=\begin{pmatrix}M_{11} & M_{12} & 0\\ M_{21} & M_{22} & 0\\ 0 & 0 & 0\end{pmatrix}\in\mathbb{R}^{3\times 3}$.

Let Ω be an open domain of $\mathbb{R}^3$. The usual Lebesgue spaces of square integrable functions, vector or tensor fields on Ω with values in $\mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{R}^{3\times 3}$, respectively, will be denoted by $\mathrm{L}^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces $\mathrm{H}^1(\Omega)=\{u\in\mathrm{L}^2(\Omega)\,|\,\mathrm{D}\,u\in\mathrm{L}^2(\Omega)\}$, $\mathrm{H}(\mathrm{curl};\Omega)=\{v\in\mathrm{L}^2(\Omega)\,|\,\mathrm{curl}\,v\in\mathrm{L}^2(\Omega)\}$ of functions u or vector fields v, respectively. For vector fields $u=(u_1,u_2,u_3)$ with $u_i\in\mathrm{H}^1(\Omega)$, $i=1,2,3$, we define $\mathrm{D}\,u:=(\mathrm{D}\,u_1\,|\,\mathrm{D}\,u_2\,|\,\mathrm{D}\,u_3)^T$, while for tensor fields P with rows in $\mathrm{H}(\mathrm{curl};\Omega)$, i.e. $P=\big(P^T.e_1\,|\,P^T.e_2\,|\,P^T.e_3\big)^T$ with $(P^T.e_i)^T\in\mathrm{H}(\mathrm{curl}\,;\Omega)$, $i=1,2,3$, we define $\mathrm{Curl}\,P:=\big(\mathrm{curl}\,(P^T.e_1)^T\,|\,\mathrm{curl}\,(P^T.e_2)^T\,|\,\mathrm{curl}\,(P^T\,e_3)^T\big)^T$. The corresponding Sobolev spaces will be denoted by $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^1(\mathrm{Curl};\Omega)$, respectively. We will use the notations: D_ξ, D_x, Curl_ξ, Curl_x, etc. to indicate the variables for which these quantities are calculated.

2.2. Geometrical nonlinear and physically linear Cosserat elastic 3D models

We consider an elastic material which in its reference configuration fills the three-dimensional domain $\Omega\subset\mathbb{R}^3$. In the Cosserat theory, each point of the reference body is endowed with three independent orthogonal directors, i.e. with a matrix field $\overline{R}:\Omega\to\mathrm{SO}(3)$ called the *microrotation* tensor. Let us remark that while the tensor $\mathrm{polar}(\mathrm{D}\varphi)\in\mathrm{SO}(3)$ of the polar decomposition of $F:=\mathrm{D}\varphi=\mathrm{polar}(\mathrm{D}\varphi)\sqrt{(\mathrm{D}\varphi)^T\mathrm{D}\varphi}$ is not independent of φ [7,36,37], the tensor $\overline{R}$ in the Cosserat theory is independent of $\mathrm{D}\varphi$. In other words, in general, $\overline{R}\neq\mathrm{polar}(\mathrm{D}\varphi)$. In geometrical nonlinear and physically linear Cosserat elastic 3D models, the deformation φ and the microrotation $\overline{R}$ are the solutions of the following *nonlinear minimisation problems* on Ω:

$$I(\varphi,F,\overline{R},\partial_{x_i}\overline{R})=\int_\Omega\Big[W_{\mathrm{strain}}(F,\overline{R})+W_{\mathrm{Cosserat-curv}}(\overline{R},\partial_{x_i}\overline{R})\Big]\,\mathrm{d}V\quad\mapsto\min.\qquad\text{w.r.t}\quad(\varphi,\overline{R}),\quad(2.1)$$

where $F=\mathrm{D}\varphi$ represents the deformation gradient, $W_{\mathrm{strain}}(F,\overline{R})$ is the strain energy, $W_{\mathrm{Cosserat-curv}}$ $(\overline{R},\partial_{x_i}\overline{R})$ is the Cosserat curvature (bending) energy and and dV denotes the volume element in the Ω-configuration. For simplicity of exposition, we consider that the external loadings are not present and we have only Dirichlet-type boundary conditions for φ.

In this paper, the strain energy is considered to be a general isotropic quadratic energy (physically linear) in terms of the non-symmetric Biot-type stretch tensor $\overline{U}:=\overline{R}^TF\in\mathbb{R}^{3\times 3}$ (the first Cosserat deformation tensor), i.e.

$$W_{\mathrm{strain}}(F,\overline{R})=W_{\mathrm{mp}}(\overline{U}):=\mu\,\|\mathrm{dev}\,\mathrm{sym}(\overline{U}-\mathbb{1}_3)\|^2+\mu_{\mathrm{c}}\,\|\mathrm{skew}(\overline{U}-\mathbb{1}_3)\|^2+\frac{\kappa}{2}\,[\mathrm{tr}(\mathrm{sym}(\overline{U}-\mathbb{1}_3))]^2\,,\quad(2.2)$$

while the Cosserat curvature (bending) energy $W_{\text{Cosserat-curv}}(\overline{R},\partial_{x_i}\overline{R})$ is considered to be isotropic in terms of $\overline{R}$ and quadratic in the following curvature strain candidates

$$\begin{aligned}
&\mathfrak{K} := \overline{R}^T \mathrm{D}\overline{R} = \Big(\overline{R}^T \partial_{x_1}\overline{R}, \overline{R}^T \partial_{x_1}\overline{R}, \overline{R}^T \partial_{x_1}\overline{R}\Big) \in \mathbb{R}^{3\times 9},\\
&\widehat{\mathfrak{K}} := \big(\overline{R}^T \mathrm{D}(\overline{R}.e_1), R^T \mathrm{D}(\overline{R}.e_2), R^T \mathrm{D}(\overline{R}.e_3)\big) \in \mathbb{R}^{3\times 9},\\
&\alpha := \overline{R}^T \operatorname{Curl}\overline{R} \in \mathbb{R}^{3\times 3} \qquad \text{(the second-order dislocation density tensor [36])}\,,\\
&\Gamma := \Big(\operatorname{axl}(\overline{R}^T \partial_{x_1}\overline{R}) \,|\, \operatorname{axl}(\overline{R}^T \partial_{x_2}\overline{R}) \,|\, \operatorname{axl}(\overline{R}^T \partial_{x_3}\overline{R})\Big) \in \mathbb{R}^{3\times 3} \text{ (thewrynesstensor)}.
\end{aligned} \tag{2.3}$$

The second-order Cosserat deformation tensor Γ (the wryness tensor) was considered as a Lagrangian strain measure for curvature orientation change [16]) since the introduction of the Cosserat model [15], the second-order dislocation density tensor α is in a direct relation to the wryness tensor via Nye's formulas and energetically controls $\mathfrak{K}$ (see [29,35]), the tensor $\mathfrak{K}$ represents a first impulse choice while $\widehat{\mathfrak{K}}$ is an ad hoc choice which is not suitable for an isotropic quadratic curvature energy. Let us notice that all the mentioned curvature tensors are frame indifferent by definition, i.e. they remain invariant after the change $\overline{R} \to \overline{Q}\,\overline{R}$, with $\overline{Q} \in \mathrm{SO}(3)$ constant. In addition, Γ, α and $\mathfrak{K}$ are isotropic, property which are is shared with $\widehat{\mathfrak{K}}$.

The suitable form of the isotropic Cosserat curvature energy is discussed in Sect. 2.3. However, let us already announce that from our point of view, the most suitable expression for analytical computations of the general isotropic energy quadratic in $\overline{R}$ is

$$\begin{aligned}
W_{\text{Cosserat-curv}}(\overline{R},\partial_{x_i}\overline{R}) = W_{\text{curv}}(\alpha) &:= \mu\, L_{\mathrm{c}}^2 \left(b_1\,\|\operatorname{sym}\alpha\|^2 + b_2\,\|\operatorname{skew}\alpha\|^2 + \frac{b_3}{4}\,[\operatorname{tr}(\alpha)]^2\right)\\
&= \mu\, L_{\mathrm{c}}^2 \left(b_1\,\|\operatorname{sym}\Gamma\|^2 + b_2\,\|\operatorname{skew}\Gamma\|^2 + b_3\,[\operatorname{tr}(\Gamma)]^2\right).
\end{aligned} \tag{2.4}$$

The parameters μ and λ are the elasticity *Lamé-type*[1] constants, $\kappa = \dfrac{2\,\mu + 3\,\lambda}{3}$ is the *infinitesimal bulk modulus*, $\mu_{\mathrm{c}} > 0$ is the *Cosserat couple modulus* and $L_c > 0$ is the *internal length* and responsible for *size effects* in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume, here, that $\mu > 0$, $\kappa > 0$, $\mu_{\mathrm{c}} > 0$. The Cosserat couple modulus μ_c controls the deviation of the microrotation $\overline{R}$ from the continuum rotation polar$(\mathrm{D}\varphi)$ in the polar decomposition of $\mathrm{D}\varphi = \operatorname{polar}(\mathrm{D}\varphi)\cdot\sqrt{\mathrm{D}\varphi^T\mathrm{D}\varphi}$. For $\mu_c \to \infty$, the constraint $R = \operatorname{polar}(\mathrm{D}\varphi)$ is generated and the model would turn into a Toupin couple stress model. We also assume that $b_1 > 0, b_2 > 0$ and $b_3 > 0$, which assures the *coercivity* and *convexity of the curvature* energy [33].

2.3. More on Cosserat curvature strain measures

2.3.1. The curvature tensor $\mathfrak{K} = \overline{R}^T\mathrm{D}\overline{R} \in \mathbb{R}^{3\times 9}$. A first choice for a curvature strain tensor is the third-order elastic Cosserat curvature tensor [12–15]

$$\mathfrak{K} := \overline{R}^T\,\mathrm{D}\,\overline{R} = \overline{R}^T\,(\partial_{x_1}\overline{R}\,|\,\partial_{x_2}\overline{R}\,|\,\partial_{x_3}\overline{R}) = (\overline{R}^T\,\partial_{x_1}\overline{R}\,|\,\overline{R}^T\,\partial_{x_2}\overline{R}\,|\,\overline{R}^T\,\partial_{x_3}\overline{R}) \in \mathbb{R}^{3\times 9}, \tag{2.5}$$

[1] In the sense that if Cosserat effects are ignored, μ and λ are the Lamé coefficients of classical isotropic elasticity.

and the curvature energy given by observing that $\widetilde{W}_{\rm curv}(\mathfrak{K}) := a_1\,\|\mathfrak{K}\|^2$. This one-parameter choice is motivated [19] by $\mathfrak{K}\equiv \operatorname{skew}\mathfrak{K}$, since $\overline{R}^T\partial_{x_i}\overline{R}\in\mathfrak{so}(3)$, $i=1,2,3$, and

$$\begin{aligned}
\operatorname{sym}\mathfrak{K} &= \big(\operatorname{sym}(\overline{R}^T\partial_{x_1}\overline{R})\,|\,\operatorname{sym}(\overline{R}^T\partial_{x_2}\overline{R})\,|\,\operatorname{sym}(\overline{R}^T\partial_{x_3}\overline{R})\big) = (0_3\,|\,0_3\,|\,0_3),\\
\operatorname{tr}\mathfrak{K} &= \operatorname{tr}(\overline{R}^T\partial_{x_1}\overline{R}) + \operatorname{tr}(\overline{R}^T\partial_{x_2}\overline{R}) + \operatorname{tr}(\overline{R}^T\partial_{x_3}\overline{R}) = 0,\\
\operatorname{skew}\mathfrak{K} &= \big(\operatorname{skew}(\overline{R}^T\partial_{x_1}\overline{R})\,|\,\operatorname{skew}(\overline{R}^T\partial_{x_2}\overline{R})\,|\,\operatorname{skew}(\overline{R}^T\partial_{x_3}\overline{R})\big).
\end{aligned}\tag{2.6}$$

However, this is not the most general form of a quadratic isotropic energy in $\overline{R}$ as will be seen later.

The third-order tensor $\mathfrak{K}=\Big(\overline{R}^T\partial_{x_1}\overline{R}\,|\,\overline{R}^T\partial_{x_2}\overline{R}\,|\,\overline{R}^T\partial_{x_3}\overline{R}\Big)\in\mathbb{R}^{3\times 9}$ is usually replaced by the wryness tensor $\Gamma=\Big(\operatorname{axl}(\overline{R}^T\partial_{x_1}\overline{R})\,|\,\operatorname{axl}(\overline{R}^T\partial_{x_2}\overline{R})\,|\,\operatorname{axl}(\overline{R}^T\partial_{x_3}\overline{R})\Big)$, since we have the one-to-one relations

$$\mathfrak{K} = \operatorname{anti}\Gamma, \qquad\qquad \Gamma = \operatorname{axl}\mathfrak{K}, \tag{2.7}$$

due to the fact that $\overline{R}^T\partial_{x_1}\overline{R}\in\mathfrak{so}(3)$, $i=1,2,3$, which in indices read

$$\mathfrak{K}_{ijk} = \overline{R}_{li}\frac{\partial \overline{R}_{lj}}{\partial x_k}, \qquad\qquad \mathfrak{K}_{ijk} = -\epsilon_{ijl}\Gamma_{lk}, \qquad\qquad \Gamma_{ik} = \frac{1}{2}\sum_{r,l=1}^{3}\epsilon_{ilr}\mathfrak{K}_{lrk}. \tag{2.8}$$

For a detailed discussion on various strain measures of the nonlinear micropolar continua, we refer to [38].

Proposition 2.1. *A general isotropic quadratic energy depending on $\overline{R}^T\mathrm{D}\overline{R}\in\mathbb{R}^{3\times 9}$ has the form*

$$\begin{aligned}
\widetilde{W}(\mathfrak{K}) &= b_1\,\|\mathit{sym}\,\operatorname{axl}\mathfrak{K}\|^2 + b_2\,\|\mathit{skew}\,\operatorname{axl}\mathfrak{K}\|^2 + 4\,b_3\,[\operatorname{tr}(\operatorname{axl}\mathfrak{K})]^2\\
&= \quad b_1\,\|\mathit{sym}\big(\operatorname{axl}(\mathfrak{K}.e_1)\,|\,\operatorname{axl}(\mathfrak{K}.e_2)\,|\,\operatorname{axl}(\mathfrak{K}.e_3)\big)\|^2\\
&\quad + b_2\,\|\mathit{skew}\big(\operatorname{axl}(\mathfrak{K}.e_1)\,|\,\operatorname{axl}(\mathfrak{K}.e_2)\,|\,\operatorname{axl}(\mathfrak{K}.e_3)\big)\|^2\\
&\quad + b_3\,[\operatorname{tr}(\big(\operatorname{axl}(\mathfrak{K}.e_1)\,|\,\operatorname{axl}(\mathfrak{K}.e_2)\,|\,\operatorname{axl}(\mathfrak{K}.e_3)\big))]^2.
\end{aligned}\tag{2.9}$$

Proof. The proof is based on the result from [17] and on the identities (2.7). Indeed, a quadratic energy in $\mathfrak{K}$ is a quadratic energy in Γ. Due to the results presented in [17], a quadratic isotropic energy written in terms of Γ is given by

$$W(\Gamma) = b_1\,\|\operatorname{sym}\Gamma\|^2 + b_2\,\|\operatorname{skew}\Gamma\|^2 + b_3\,[\operatorname{tr}(\Gamma)]^2. \tag{2.10}$$

Using (2.7), the proof is complete. □

We can express the uni-constant isotropic curvature term as a positive definite quadratic form in terms of Γ, i.e.

$$\begin{aligned}
\|\mathrm{D}\overline{R}\|^2_{\mathbb{R}^{3\times3\times3}} &= \|\overline{R}^T\mathrm{D}\overline{R}\,\|^2_{\mathbb{R}^{3\times3\times3}} = \|\mathfrak{K}\|^2_{\mathbb{R}^{3\times3\times3}} = \|\overline{R}^T\partial_{x_1}\overline{R}\|^2_{\mathbb{R}^{3\times3}} + \|\overline{R}^T\partial_{x_2}\overline{R}\|^2_{\mathbb{R}^{3\times3}} + \|\overline{R}^T\partial_{x_3}\overline{R}\|^2_{\mathbb{R}^{3\times3}}\\
&= 2\,\|\operatorname{axl}(\overline{R}^T\partial_{x_1}\overline{R})\|^2_{\mathbb{R}^3} + 2\,\|\operatorname{axl}(\overline{R}^T\partial_{x_2}\overline{R})\|^2_{\mathbb{R}^3}2\,\|\operatorname{axl}(\overline{R}^T\partial_{x_3}\overline{R})\|^2_{\mathbb{R}^3} = 2\,\|\Gamma\|^2_{\mathbb{R}^{3\times3}}.
\end{aligned}\tag{2.11}$$

Therefore, a general positive definite quadratic isotropic curvature energy (2.9) in Γ is a positive definite quadratic form in terms of $\|\overline{R}^T\mathrm{D}\overline{R}\|^2_{\mathbb{R}^{3\times3\times3}}$, and vice versa. Thus, working with a quadratic isotropic positive definite energy in terms of $\overline{R}^T\mathrm{D}\overline{R}$ is equivalent with working with a quadratic positive definite isotropic energy in terms of Γ. Since the expression of a isotropic curvature energy has a simpler form in terms of Γ, in order to keep the calculations as simple as possible, we prefer to work with Γ.

2.3.2. The curvature tensor $\boldsymbol{\alpha} = \overline{\boldsymbol{R}}^T \,\mathbf{Curl}\, \overline{\boldsymbol{R}} \in \mathbb{R}^{3\times 3}$. Another choice as curvature strain is the *second-order dislocation density tensor* α. Comparing with $\overline{R}^T \mathrm{D}\, \overline{R}$, it simplifies considerably the representation by allowing to use the orthogonal decomposition

$$\overline{R}^T \,\mathrm{Curl}\, \overline{R} = \alpha = \operatorname{dev}\operatorname{sym}\alpha + \operatorname{skew}\alpha + \frac{1}{3}\operatorname{tr}(\alpha)\mathbb{1}_3. \tag{2.12}$$

Moreover, it yields an equivalent control of spatial derivatives of rotations [35] and allows us to write the curvature energy in a fictitious Cartesian configuration in terms of the wryness tensor [16,35] $\Gamma \in \mathbb{R}^{3\times 3}$, since (see [35]) the following close relationship between the *wryness tensor* and the *dislocation density tensor* holds

$$\alpha = -\Gamma^T + \operatorname{tr}(\Gamma)\,\mathbb{1}_3, \qquad \text{or equivalently,} \qquad \Gamma = -\alpha^T + \frac{1}{2}\operatorname{tr}(\alpha)\,\mathbb{1}_3. \tag{2.13}$$

Hence,

$$\begin{aligned} \operatorname{sym}\Gamma &= -\operatorname{sym}\alpha + \frac{1}{2}\operatorname{tr}(\alpha)\,\mathbb{1}_3, & \operatorname{dev}\operatorname{sym}\Gamma &= -\operatorname{dev}\operatorname{sym}\alpha, \\ \operatorname{skew}\Gamma &= -\operatorname{skew}(\alpha^T) = \operatorname{skew}\alpha, & \operatorname{tr}(\Gamma) &= -\operatorname{tr}(\alpha) + \frac{3}{2}\operatorname{tr}(\alpha) = \frac{1}{2}\operatorname{tr}(\alpha) \end{aligned} \tag{2.14}$$

and

$$\operatorname{sym}\alpha = -\operatorname{sym}\Gamma + \operatorname{tr}(\Gamma)\,\mathbb{1}_3, \quad \operatorname{dev}\operatorname{sym}\alpha = -\operatorname{dev}\operatorname{sym}\Gamma, \quad \operatorname{skew}\alpha = \operatorname{skew}\Gamma, \quad \operatorname{tr}(\alpha) = 2\operatorname{tr}(\Gamma). \tag{2.15}$$

In addition, from [17] we have

Proposition 2.2. *A general quadratic isotropic energy depending on α has the form*

$$W_{\mathrm{curv}}(\alpha) = b_1\,\|sym\,\alpha\|^2 + b_2\,\|skew\,\alpha\|^2 + \frac{b_3}{4}\,[\operatorname{tr}(\alpha)]^2. \tag{2.16}$$

Proof. We use again that a quadratic energy in α is a quadratic energy in Γ, i.e. due to [17], is given by (2.10). The proof is complete after using the Nye's formulas (2.13). □

Since a quadratic isotropic positive definite energy in terms of $\overline{R}^T \mathrm{D}\overline{R}$ is equivalent with a quadratic positive definite isotropic energy in terms of Γ, considering α is equivalent with considering $\mathfrak{K}$, as long as a quadratic isotropic energy is used. As we will see in the present paper, a quadratic curvature energy in terms of α is suitable for explicit calculations of homogenised curvature energy for shell models via the Γ-convergence method.

2.3.3. The curvature (in fact: bending) tensor $\widehat{\boldsymbol{\mathfrak{K}}} = \big(\overline{\boldsymbol{R}}^T \mathbf{D}(\overline{\boldsymbol{R}}.\boldsymbol{e_1})\,|\,\overline{\boldsymbol{R}}^T \mathbf{D}(\overline{\boldsymbol{R}}.\boldsymbol{e_2})\,|\,\overline{\boldsymbol{R}}^T \mathbf{D}(\overline{\boldsymbol{R}}.\boldsymbol{e_3})\big) \in \mathbb{R}^{3\times 9}$. The curvature tensor $\widehat{\mathfrak{K}} = \big(\overline{R}^T \mathrm{D}(\overline{R}.e_1)\,|\,\overline{R}^T \mathrm{D}(\overline{R}.e_2)\,|\,\overline{R}^T \mathrm{D}(\overline{R}.e_3)\big) \in \mathbb{R}^{3\times 9}$ is motivated from the flat Cosserat shell model [31,32]. Indeed, in this setting, the general bending energy term appearing from an engineering ansatz through the thickness of the shell appears as

$$\frac{h^3}{12}\left(\mu\,\|\operatorname{sym}(\overline{R}^T \mathrm{D}(\overline{R}.e_3))\|^2 + \mu_c\,\|\operatorname{skew}(\overline{R}^T \mathrm{D}(\overline{R}.e_3))\|^2 + \frac{\lambda\,\mu}{\lambda + 2\mu}\left[\operatorname{tr}(\overline{R}^T \mathrm{D}(\overline{R}.e_3))\right]^2\right). \tag{2.17}$$

Motivated by this, in earlier papers [31–34], as generalisation of $\overline{R}^T \mathrm{D}(\overline{R}.e_3)$, the third-order elastic Cosserat curvature tensor is considered in the form

$$\widehat{\mathfrak{K}} = \big(\overline{R}^T \mathrm{D}(\overline{R}.e_1)\,|\,\overline{R}^T \mathrm{D}(\overline{R}.e_2)\,|\,\overline{R}^T \mathrm{D}(\overline{R}.e_3)\big) = \overline{R}^T \big(\mathrm{D}(\overline{R}.e_1), \mathrm{D}(\overline{R}.e_2), \mathrm{D}(\overline{R}.e_3)\big) \in \mathbb{R}^{3\times 9}, \tag{2.18}$$

treating the three directions e_1, e_2, e_3 equally, and the curvature energy is taken to be

$$\widehat{W}_{\mathrm{curv}}(\widehat{\mathfrak{K}}) = a_1\|\operatorname{sym}\widehat{\mathfrak{K}}\|^2 + a_2\|\operatorname{skew}\widehat{\mathfrak{K}}\|^2 + a_3[\operatorname{tr}(\widehat{\mathfrak{K}})]^2. \tag{2.19}$$

We mean by $\widehat{\mathfrak{K}}.e_i = \overline{R}^T \mathrm{D}(\overline{R}.e_i)$ and

$$\begin{aligned}
\|\mathrm{sym}\,\widehat{\mathfrak{K}}\|^2 &= \sum_{i=1}^{3} \|\mathrm{sym}\,\widehat{\mathfrak{K}}.e_i\|^2 = \sum_{i=1}^{3} \|\mathrm{sym}\,(\overline{R}^T \mathrm{D}(\overline{R}.e_i))\|^2,\\
\|\mathrm{skew}\,\widehat{\mathfrak{K}}\|^2 &= \sum_{i=1}^{3} \|\mathrm{skew}\,\widehat{\mathfrak{K}}.e_i\|^2 = \sum_{i=1}^{3} \|\mathrm{skew}\,(\overline{R}^T \mathrm{D}(\overline{R}.e_i))\|^2,\\
[\mathrm{tr}(\widehat{\mathfrak{K}})]^2 &= \sum_{i=1}^{3} [\mathrm{tr}(\widehat{\mathfrak{K}}.e_i)]^2 = \sum_{i=1}^{3} [\mathrm{tr}(\overline{R}^T \mathrm{D}(\overline{R}.e_i))]^2.
\end{aligned} \tag{2.20}$$

However, this curvature energy has now three abstract orthogonal preferred directions, which makes it only cubic and not isotropic, as we will see.

There does not exist an analysis to show that this is the most general form of an isotropic energy depending on $\widehat{\mathfrak{K}}$. Actually, as we will see in the following, for general positive values for $(\alpha_i^1, \alpha_i^2, \alpha_i^3)$ the energies of the form (2.17) are anisotropic. We simplify the discussion by considering only the energy $\|\overline{R}^T \mathrm{D}(\overline{R}.e_i)\|^2$. After the transformation $\overline{R} \to \overline{R}\,\overline{Q}$, with $\overline{Q} = (e_1\,|\,e_3\,|\,e_2) \in \mathrm{SO}(3)$ constant, we have

$$\|\overline{Q}^T \overline{R}^T \mathrm{D}(\overline{R}\,\overline{Q}.e_3)\|^2 = \|\overline{R}^T \mathrm{D}(\overline{R}\,\overline{Q}.e_3)\|^2 = \|\overline{R}^T \mathrm{D}(\overline{R}.e_2)\|^2 \neq \|\overline{R}^T \mathrm{D}(\overline{R}.e_3)\|^2. \tag{2.21}$$

As regards the direct relation between $\|\widehat{\mathfrak{K}}\|^2 = \sum_{i=1}^3 \left\|\overline{R}^T \mathrm{D}(\overline{R}.e_i)\right\|^2$ and $\|\alpha\|^2$ we have

$$\begin{aligned}
\sum_{i=1}^{3} \left\|\overline{R}^T \mathrm{D}(\overline{R}.e_i)\right\|^2_{\mathbb{R}^{3\times3}} &= \sum_{i=1}^{3} \left\|\mathrm{D}(\overline{R}.e_i)\right\|^2_{\mathbb{R}^{3\times3}} = \left\|\mathrm{D}\overline{R}\right\|^2_{\mathbb{R}^{3\times3}}\\
&= 1\cdot\|\mathrm{dev}\,\mathrm{sym}\overline{R}^T\mathrm{Curl}\overline{R}\|^2_{\mathbb{R}^{3\times3}} + 1\cdot\|\mathrm{skew}\overline{R}^T\mathrm{Curl}\overline{R}\|^2_{\mathbb{R}^{3\times3}} + \frac{1}{12}\cdot[\mathrm{tr}(\overline{R}^T\mathrm{Curl}\overline{R})]^2\\
&\geq c_+\left\|\mathrm{Curl}\overline{R}\,\right\|^2_{\mathbb{R}^{3\times3}},
\end{aligned} \tag{2.22}$$

where $c_+ > 0$ is a constant. Since a coercive curvature energy in $\widehat{\mathfrak{K}}$ is completely controlled by $\sum_{i=1}^3 \left\|\overline{R}^T \mathrm{D}(\overline{R}.e_i)\right\|^2_{\mathbb{R}^{3\times3}}$, a positive definite quadratic isotropic energy of the form in terms of $\alpha = \overline{R}^T\mathrm{Curl}\overline{R}$ (equivalently on the wryness tensor Γ) is a positive definite quadratic form in terms of $\|\overline{R}^T\mathrm{D}\overline{R}\|^2_{\mathbb{R}^{3\times3\times3}}$, and vice versa. Hence, a quadratic positive definite energy in terms of $\widehat{\mathfrak{K}}$ is energetically equivalent with a quadratic positive definite energy in terms of α (and Γ).

Let us remark that both $\left(\partial_{x_1}\overline{R}\,|\,\partial_{x_2}\overline{R}\,|\,\partial_{x_3}\overline{R}\right) \in \mathbb{R}^{3\times9}$, $\left(\mathrm{D}(\overline{R}.e_1), \mathrm{D}(\overline{R}.e_2), \mathrm{D}(\overline{R}.e_3)\right)$ contain the same terms, $\frac{\partial \overline{R}_{ij}}{\partial x_k}$, $i,j,k = 1,2,3$, but differently ordered. By multiplication with $\overline{R}^T$ of $\left(\partial_{x_1}\overline{R}\,|\,\partial_{x_2}\overline{R}\,|\,\partial_{x_3}\overline{R}\right) \in \mathbb{R}^{3\times9}$ and $\left(\mathrm{D}(\overline{R}.e_1)\,|\,\mathrm{D}(\overline{R}.e_2)\,|\,\mathrm{D}(\overline{R}.e_3)\right)$ we obtain $\mathfrak{K}$ and $\widehat{\mathfrak{K}}$, respectively, i.e. $\mathfrak{K}_{ijk} = \overline{R}_{li}\frac{\partial \overline{R}_{lj}}{\partial x_k}$, $\widehat{\mathfrak{K}}_{ijk} = \overline{R}_{li}\frac{\partial \overline{R}_{lk}}{\partial x_j}, i,j,k = 1,2,3$ and $\mathfrak{K}_{ijk} = \widehat{\mathfrak{K}}_{ikj},\ i,j,k = 1,2,3$. We have the following relation between $\widehat{\mathfrak{K}}$ and Γ

$$\widehat{\mathfrak{K}}_{ijk} = \mathfrak{K}_{ikj} = -\epsilon_{ikl}\Gamma_{lj}, \qquad \Gamma_{ik} = \frac{1}{2}\sum_{r,l=1}^{3}\epsilon_{ilr}\mathfrak{K}_{lrk} = \frac{1}{2}\sum_{r,l=1}^{3}\epsilon_{ilr}\widehat{\mathfrak{K}}_{lkr}. \tag{2.23}$$

Let us introduce the operator $\mathcal{A}: \mathbb{R}^{3\times9} \to \mathbb{R}^{3\times9}$ by $(\mathcal{A}.\widehat{\mathfrak{K}})_{ijk} = \widehat{\mathfrak{K}}_{ikj}$.

Proposition 2.3. *A general isotropic energy depending on* $\widehat{\mathfrak{K}} = \left(\overline{R}^T\mathrm{D}(\overline{R}.e_1)\,|\,\overline{R}^T\mathrm{D}(\overline{R}.e_2)\,|\,\overline{R}^T\mathrm{D}(\overline{R}.e_3)\right) \in \mathbb{R}^{3\times9}$ *has the form*

$$\widetilde{W}(\widehat{\mathfrak{K}}) = b_1\,\|sym\;\mathrm{axl}(\mathcal{A}.\widehat{\mathfrak{K}})\|^2 + b_2\,\|skew\;\mathrm{axl}(\mathcal{A}.\widehat{\mathfrak{K}})\|^2 + b_3\,[\mathrm{tr}(\mathrm{axl}(\mathcal{A}.\widehat{\mathfrak{K}}))]^2. \tag{2.24}$$

Proof. Using (2.1), we have that a quadratic isotropic energy in $\widehat{\mathfrak{K}}$ is given by

$$\widetilde{W}(\widehat{\mathfrak{K}}) = b_1\, \|\text{sym axl}\, \mathfrak{K}\|^2 + b_2\, \|\text{skew axl}\, \mathfrak{K}\|^2 + b_3\, [\text{tr}(\text{axl}\, \mathfrak{K})]^2. \tag{2.25}$$

Since $\mathcal{A}.\widehat{\mathfrak{K}} = \mathfrak{K}$, the proof is complete. □

Let us remark that comparing to (2.17), a general isotropic energy depending on $\widehat{\mathfrak{K}}$, i.e. (2.24) is different since we have the summation of different products between the elements of $\overline{R}^T$ and $\mathrm{D}\overline{R}$, due to the action of the axial operator together with the operator $\mathcal{A}$.

From (2.17), one could obtain an isotropic energy by setting

$$\int_{\widetilde{Q}\in\mathrm{SO}(3)} \frac{h^3}{12}\sum_{i=1}^{3}\left(\mu\, \|\,\text{sym}(\overline{R}^T\mathrm{D}(\widetilde{Q}.e_i))\|^2 + \mu_{\mathrm{c}}\, \|\text{skew}(\overline{R}^T\mathrm{D}(\widetilde{Q}.e_i))\|^2 + \frac{\lambda\,\mu}{\lambda+2\mu}\left[\text{tr}(\overline{R}^T\mathrm{D}(\widetilde{Q}.e_i))\right]^2\right), \tag{2.26}$$

i.e. averaging over all directions.

3. Homogenised curvature energy for the flat Cosserat shell model via Γ-convergence

Let us consider an elastic material which in its reference configuration fills the three-dimensional *flat shell-like thin* domain $\Omega_h = \omega \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ and $\omega \subset \mathbb{R}^2$ a bounded domain with Lipschitz boundary $\partial\omega$. The scalar $0 < h \ll 1$ is called *thickness* of the shell.

Due to the discussion from Sect. 2.3, in this paper we consider the Cosserat curvature energy in terms of the wryness tensor Γ in the form

$$\widetilde{W}_{\text{curv}}(\Gamma) \;=\; \mu\, L_{\mathrm{c}}^2\left(b_1\, \|\text{sym}\,\Gamma\|^2 + b_2\, \|\text{skew}\,\Gamma\|^2 +\, b_3\, [\text{tr}(\Gamma)]^2\right)\,. \tag{3.1}$$

In order to apply the methods of Γ-convergence for constructing the variational problem on ω of the flat Cosserat shell model, the first step is to transform our problem further from Ω_h to a *domain* with fixed thickness $\Omega_1 = \omega \times [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}^3$, $\omega \subset \mathbb{R}^2$. For this goal, the scaling of the variables (dependent/independent) would be the first step. In all our computations, the mark $\cdot^\natural$ indicates the nonlinear scaling and the mark $\cdot_h$ indicates that the assigned quantity depends on the thickness h. In a first step, we will apply the nonlinear scaling to the deformation. For $\Omega_1 = \omega \times \left[-\dfrac{1}{2}, \dfrac{1}{2}\right] \subset \mathbb{R}^3$, $\omega \subset \mathbb{R}^2$, we define the scaling transformations

$$\zeta\colon\ \eta\in\Omega_1 \mapsto \mathbb{R}^3\,,\qquad \zeta(\eta_1,\eta_2,\eta_3) := (\eta_1,\eta_2,h\,\eta_3)\,,\quad \zeta^{-1}\colon\ x\in\Omega_h\mapsto\mathbb{R}^3\,,\qquad \zeta^{-1}(x_1,x_2,x_3) := (x_1,x_2,\frac{x_3}{h})\,,$$

with $\zeta(\Omega_1) = \Omega_h$. By using the above transformations (3.2), we obtain the formula for the transformed deformation φ as

$$\varphi(x_1,x_2,x_3) = \varphi^\natural(\zeta^{-1}(x_1,x_2,x_3)) \quad \forall x\in\Omega_h\,;\qquad \varphi^\natural(\eta) = \varphi(\zeta(\eta)) \quad \forall \eta\in\Omega_1\,,$$

$$\mathrm{D}_x\varphi(x_1,x_2,x_3) = \begin{pmatrix} \partial_{\eta_1}\varphi_1^\natural(\eta) & \partial_{\eta_2}\varphi_1^\natural(\eta) & \frac{1}{h}\partial_{\eta_3}\varphi_1^\natural(\eta)\\ \partial_{\eta_1}\varphi_2^\natural(\eta) & \partial_{\eta_2}\varphi_2^\natural(\eta) & \frac{1}{h}\partial_{\eta_3}\varphi_2^\natural(\eta)\\ \partial_{\eta_1}\varphi_3^\natural(\eta) & \partial_{\eta_2}\varphi_3^\natural(\eta) & \frac{1}{h}\partial_{\eta_3}\varphi_3^\natural(\eta) \end{pmatrix} = \mathrm{D}_\eta^h\varphi^\natural(\eta) =: F_h^\natural\,. \tag{3.2}$$

Now we will do the same process for the microrotation tensor $\overline{R}_h^\natural\colon \Omega_1 \to \mathrm{SO}(3)$

$$\overline{R}(x_1,x_2,x_3) = \overline{R}_h^\natural(\zeta^{-1}(x_1,x_2,x_3)) \qquad \forall x\in\Omega_h\,;\ \ \overline{R}_h^\natural(\eta) = \overline{R}(\zeta(\eta))\,,\quad \forall\eta\in\Omega_1\,. \tag{3.3}$$

With this, the non-symmetric stretch tensor expressed in a point of Ω_1 is given by

$$\overline{U}_e^\natural = \overline{R}_h^{\natural,T} F_h^\natural = \overline{R}_h^{\natural,T} \mathrm{D}_\eta^h \varphi^\natural(\eta)\,. \tag{3.4}$$

and

$$\Gamma^\natural_{e,h} = \Big(\mathrm{axl}(\overline{R}_h^{\natural,T}\,\partial_{\eta_1}\overline{R}_h^\natural)\,|\,\mathrm{axl}(\overline{R}_h^{\natural,T}\,\partial_{\eta_2}\overline{R}_h^\natural)\,|\,\frac{1}{h}\mathrm{axl}(\overline{R}_h^{\natural,T}\,\partial_{\eta_3}\overline{R}_h^\natural)\Big). \tag{3.5}$$

The next step, in order to apply the Γ-convergence technique, is to transform the minimisation problem onto the *fixed domain* Ω_1, which is independent of the thickness h. According to the results from the previous subsection, we have found that the original three-dimensional variational problem (2.3) is equivalent to the following minimisation problem on Ω_1

$$I_h^\natural(\varphi^\natural,\mathrm{D}_\eta^h\varphi^\natural,\overline{R}_h^\natural,\Gamma^\natural_{e,h}) = \int_{\Omega_1} h\,\Big[\Big(W_{\mathrm{mp}}(U^\natural_{e,h}) + \widetilde{W}_{\mathrm{curv}}(\Gamma^\natural_{e,h})\Big)\Big]\,\mathrm{d}V_\eta \quad \mapsto \quad \min \text{ w.r.t } (\varphi^\natural,\overline{R}_h^\natural)\,, \tag{3.6}$$

where

$$\begin{aligned}
W_{\mathrm{mp}}(U^\natural_{e,h}) =&\ \mu\,\|\mathrm{sym}(U^\natural_{e,h}-\mathbb{1}_3)\|^2 + \mu_c\,\|\mathrm{skew}(U^\natural_{e,h}-\mathbb{1}_3)\|^2 + \frac{\lambda}{2}[\mathrm{tr}(\mathrm{sym}(U^\natural_{e,h}-\mathbb{1}_3))]^2\,,\\
\widetilde{W}_{\mathrm{curv}}(\Gamma^\natural_{e,h}) =&\ \mu\,L_{\mathrm{c}}^2\,\Big(a_1\,\|\mathrm{dev}\,\mathrm{sym}\,\Gamma^\natural_{e,h}\|^2 + a_2\,\|\mathrm{skew}\,\Gamma^\natural_{e,h}\|^2 + a_3\,[\mathrm{tr}(\Gamma^\natural_{e,h})]^2\Big)\\
=&\ \mu\,L_{\mathrm{c}}^2\,\Big(b_1\,\|\mathrm{sym}\,\Gamma^\natural_{e,h}\|^2 + b_2\,\|\mathrm{skew}\,\Gamma^\natural_{e,h}\|^2 + b_3\,[\mathrm{tr}(\Gamma^\natural_{e,h})]^2\Big)\,,
\end{aligned} \tag{3.7}$$

where $a_1 = b_1$, $a_2 = b_2$ and $a_3 = \frac{12b_3 - b_1}{3}$.

In article [33], one aim of the authors was to to find the Γ-limit of the family of functional which is related to

$$\mathcal{I}_h^\natural(\varphi^\natural,\mathrm{D}_\eta^h\varphi^\natural,\overline{R}_h^\natural,\Gamma^\natural_h) = \begin{cases} \dfrac{1}{h}\,I_h^\natural(\varphi^\natural,\mathrm{D}_\eta^h\varphi^\natural,\overline{R}_h^\natural,\Gamma^\natural_h) & \text{if } (\varphi^\natural,\overline{R}_h^\natural)\in\mathcal{S}',\\ +\infty & \text{else in } X, \end{cases} \tag{3.8}$$

where

$$\begin{aligned}
X &:= \{(\varphi^\natural,\overline{R}_h^\natural)\in \mathrm{L}^2(\Omega_1,\mathbb{R}^3)\times\mathrm{L}^2(\Omega_1,\mathrm{SO}(3))\}\,,\\
\mathcal{S}' &:= \{(\varphi,\overline{R}_h)\in \mathrm{H}^1(\Omega_1,\mathbb{R}^3)\times\mathrm{H}^1(\Omega_1,\mathrm{SO}(3))\ \big|\ \varphi|_{\partial\Omega_1}(\eta) = \varphi_d^\natural(\eta)\}\,.
\end{aligned} \tag{3.9}$$

That means, to obtain an energy functional expressed only in terms of the weak limit of a subsequence of $(\varphi^\natural_{h_j},\overline{R}^\natural_{h_j})\in X$, when h_j goes to zero. In other words, as we will see, to construct an energy function depending only on quantities definite on the planar midsurface ω. However, in [33] the authors have considered a different Cosserat curvature energy based on the Cosserat curvature tensor $\widehat{\mathfrak{K}} = (\overline{R}^T\mathrm{D}(\overline{R}.e_1)R_1, \overline{R}^T\mathrm{D}(\overline{R}.e_2), \overline{R}^T\mathrm{D}(\overline{R}.e_3))\in\mathbb{R}^{3\times3\times3}$, which in the simplest form reads

$$\widehat{W}_{\mathrm{curv}}(\widehat{\mathfrak{K}}) = \mu\frac{L_c^2}{12}\Big(\alpha_1\|\mathrm{sym}\widehat{\mathfrak{K}}\|^2 + \alpha_2\|\mathrm{skew}\widehat{\mathfrak{K}}\|^2 + \alpha_3\mathrm{tr}(\widehat{\mathfrak{K}})^2\Big)\,, \tag{3.10}$$

and no explicit form of the homogenised curvature energy has been computed (perhaps even not possible to be computed for curved initial configuration). In fact, $\widehat{\mathfrak{K}}$ is not isotropic and has to be avoided in an isotropic model, as seen above.

In order to construct the Γ-limit, there is the need to solve two auxiliary optimisation problems, i.e.

O1: The optimisation problem which for each pair $(m,\overline{R}_0)$, where $m:\omega\to\mathbb{R}^3$, $\overline{R}_0:\omega\to\mathrm{SO}(3)$ defines the homogenised membrane energy

$$W_{\mathrm{mp}}^{\mathrm{hom,plate}}(\mathcal{E}^{\mathrm{plate}}_{m,\overline{R}_0}) := \inf_{\widetilde{d}\in\mathbb{R}^3} W_{\mathrm{mp}}\Big(\overline{R}_0^T(\mathrm{D}m|\widetilde{d})\Big) = \inf_{\widetilde{d}\in\mathbb{R}^3} W_{\mathrm{mp}}\Big(\mathcal{E}^{\mathrm{plate}}_{m,\overline{R}_0} - (0|0|\widetilde{d})\Big). \tag{3.11}$$

where $\mathcal{E}^{\mathrm{plate}}_{m,\overline{R}_0} := \overline{R}_0^T(\mathrm{D}m|0) - \mathbb{1}_2^\flat$ denotes the *elastic strain tensor* for the flat Cosserat shell model.

O2: The optimisation problem which for each $\overline{R}_0 : \omega \to \mathrm{SO}(3)$ defines the homogenised curvature energy

$$\widetilde{W}^{\text{hom,plate}}_{\text{curv}}(\mathcal{K}^{\text{plate}}_{\overline{R}_0}) := \widetilde{W}^{*}_{\text{curv}}\Big(\operatorname{axl}(\overline{R}_0^T\,\partial_{\eta_1}\overline{R}_0)\,|\,\operatorname{axl}(\overline{R}_0^T\,\partial_{\eta_2}\overline{R}_0)\,|\,\operatorname{axl}(A^*)\Big)$$
$$= \inf_{A\in\mathfrak{so}(3)} \widetilde{W}_{\text{curv}}\Big(\operatorname{axl}(\overline{R}_0^T\,\partial_{\eta_1}\overline{R}_0)\,|\,\operatorname{axl}(\overline{R}_0^T\,\partial_{\eta_2}\overline{R}_0)\,|\,\operatorname{axl}(A)\Big) \tag{3.12}$$

where $\mathcal{K}^{\text{plate}}_{\overline{R}_0} := \Big(\operatorname{axl}(\overline{R}_0^T\,\partial_{x_1}\overline{R}_0)\,|\,\operatorname{axl}(\overline{R}_0^T\,\partial_{x_2}\overline{R}_0)\,|0\Big) \notin \mathrm{Sym}(3)$ denotes the elastic bending curvature tensor for the flat Cosserat shell model.

The first optimisation problem O1 was solved in [33] giving

$$W^{\text{hom,plate}}_{\text{mp}}(\mathcal{E}^{\text{plate}}_{m,\overline{R}_0}) = W_{\text{shell}}\big([\mathcal{E}^{\text{plate}}_{m,\overline{Q}_{e,0}}]^{\parallel}\big) + \frac{2\,\mu\,\mu_c}{\mu_c+\mu}\|[\mathcal{E}^{\text{plate}}_{m,\overline{Q}_{e,0}}]^{\perp}\|^2,$$

with the orthogonal decomposition in the tangential plane and in the normal direction[2]

$$[\mathcal{E}^{\text{plate}}_{m,\overline{R}_0}]^{\parallel} := (\mathbb{1}_3 - e_3\otimes e_3)\,[\mathcal{E}^{\text{plate}}_{m,\overline{R}_0}], \qquad [\mathcal{E}^{\text{plate}}_{m,\overline{R}_0}]^{\perp} := (e_3\otimes e_3)\,[\mathcal{E}^{\text{plate}}_{m,\overline{R}_0}] \tag{3.14}$$

and

$$W_{\text{shell}}\big([\mathcal{E}^{\text{plate}}_{m,\overline{R}_0}]^{\parallel}\big) = \mu\,\|\text{sym}\,[\mathcal{E}^{\text{plate}}_{m,\overline{R}_0}]^{\parallel}\|^2 + \mu_c\,\|\text{skew}\,[\mathcal{E}^{\text{plate}}_{m,}]^{\parallel}\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\big[\operatorname{tr}([\mathcal{E}^{\text{plate}}_{m,\overline{R}_0}]^{\parallel})\big]^2. \tag{3.15}$$

As regards the second optimisation problem O2, in [33] the authors had to solve a similar problem but corresponding to a curvature energy given by (3.10), i.e. the dimensionally reduced homogenised curvature energy is defined through the

$$W^{\text{hom, plate}}_{\text{curv}}(\mathcal{A}) = \inf_{u,v,w\in\mathbb{R}^3} \widehat{W}_{\text{curv}}\Big((\mathcal{A}e_1|u),(\mathcal{A}e_2|v),(\mathcal{A}e_3|w)\Big), \tag{3.16}$$

where $\mathcal{A} := (\overline{R}_0^T(\partial_{x_1}(\overline{R}_0e_1)|\partial_{x_2}(\overline{R}_0e_1)),\overline{R}_0^T(\partial_{x_1}(\overline{R}_0)e_2|\partial_{x_2}(\overline{R}_0e_2)),\overline{R}_0^T(\partial_{x_1}(\overline{R}_0e_3)|\partial_{x_2}(\overline{R}_0e_3)))$. In this representation, calculating the homogenised energy looks more difficult and it was not explicitly done.

In this section, we show that considering the curvature energy depending on the Cosserat curvature tensor α (equivalently on the three-dimensional wryness tensor Γ) the calculation of the homogenised curvature energy (i.e. the solution of O2) is easier and analytically achievable.

Theorem 3.1. *The homogenised curvature energy for a flat Cosserat shell model is given by*

$$W^{\text{hom,plate}}_{\text{curv}}(\Gamma) = \mu L_c^2\Big(b_1\|\text{sym}[\mathcal{K}^{\text{plate}}_{\overline{R}_0}]^{\parallel}\|^2 + b_2\|\text{skew}[\mathcal{K}^{\text{plate}}_{\overline{R}_0}]^{\parallel}\|^2 + \frac{b_1b_3}{(b_1+b_3)}\operatorname{tr}([\mathcal{K}^{\text{plate}}_{\overline{R}_0}]^{\parallel})^2 + \frac{2\,b_1b_2}{b_1+b_2}\|[\mathcal{K}^{\text{plate}}_{\overline{R}_0}]^{\perp}\|\Big)$$

with the orthogonal decomposition in the tangential plane and in the normal direction

$$\mathcal{K}^{\text{plate}}_{\overline{R}_0} = [\mathcal{K}^{\text{plate}}_{\overline{R}_0}]^{\parallel} + [\mathcal{K}^{\text{plate}}_{\overline{R}_0}]^{\perp}, \qquad [\mathcal{K}^{\text{plate}}_{\overline{R}_0}]^{\parallel} := e_3\otimes e_3\,\mathcal{K}^{\text{plate}}_{\overline{R}_0}, \qquad [\mathcal{K}^{\text{plate}}_{\overline{R}_0}]^{\perp} := (\mathbb{1}_3 - e_3\otimes e_3)\,\mathcal{K}^{\text{plate}}_{\overline{R}_0}. \tag{3.17}$$

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[2] Here, for vectors $\xi,\eta\in\mathbb{R}^n$, we have considered the tensor product $(\xi\otimes\eta)_{ij} = \xi_i\,\eta_j$. Let us denote by $\overline{R}_i$ the columns of the matrix $\overline{R}$, i.e. $\overline{R} = (\overline{R}_1\,|\,\overline{R}_2\,|\,\overline{R}_3)$, $\overline{R}_i = \overline{R}\,e_i$. Since $(\mathbb{1}_3 - e_3\otimes e_3)\overline{R}^T = (\overline{R}_1\,|\,\overline{R}_2\,|\,0)^T$, it follows that $[\mathcal{E}^{\text{plate}}_{m,\overline{Q}_{e,0}}]^{\parallel} = (\overline{R}_1\,|\,\overline{R}_2\,|\,0)^T(\mathrm{D}m|0) - \mathbb{1}_2^\flat = ((\overline{R}_1\,|\,\overline{R}_2)^T\,\mathrm{D}m)^\flat - \mathbb{1}_2^\flat$, while

$$[\mathcal{E}^{\text{plate}}_{m,\overline{Q}_{e,0}}]^{\perp} = (0\,|\,0\,|\,\overline{R}_3)^T(\mathrm{D}m|0) = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ \langle\overline{R}_3,\partial_{x_1}m\rangle & \langle\overline{R}_3,\partial_{x_2}m\rangle & 0\end{pmatrix}. \tag{3.13}$$

Proof. Let us define $\Gamma_0 = (\Gamma^0_1\,|\,\Gamma^0_2\,|\,\Gamma^0_3) := \Big(\mathrm{axl}(\overline{R}_0^T\,\partial_{\eta_1}\overline{R}_0)\,|\,\mathrm{axl}(\overline{R}_0^T\,\partial_{\eta_2}\overline{R}_0)\,|\,\mathrm{axl}\,(A)\Big)$. Then the homogenised curvature energy turns out to be

$$W^{\mathrm{hom}}_{\mathrm{curv}}((\Gamma^0_1\,|\,\Gamma^0_2)) = \widetilde{W}_{\mathrm{curv}}(\Gamma^0_1\,|\,\Gamma^0_2|c^*) = \inf_{c\in\mathbb{R}^3} W_{\mathrm{curv}}((\Gamma^0_1\,|\,\Gamma^0_2|c))\,. \tag{3.18}$$

By using the relation (3.1), we start to do the calculations for the sym, skew and trace parts as

$$\mathrm{sym}\Gamma^0 = \begin{pmatrix} \Gamma^0_{11} & \frac{\Gamma^0_{12}+\Gamma^0_{21}}{2} & \frac{c_1+\Gamma^0_{31}}{2} \\ \frac{\Gamma^0_{21}+\Gamma^0_{12}}{2} & \Gamma^0_{22} & \frac{c_2+\Gamma^0_{32}}{2} \\ \frac{\Gamma^0_{31}+c_1}{2} & \frac{\Gamma^0_{32}+c_2}{2} & c_3 \end{pmatrix}, \qquad \mathrm{skew}\,\Gamma^0 = \begin{pmatrix} 0 & \frac{\Gamma^0_{12}-\Gamma^0_{21}}{2} & \frac{c_1-\Gamma^0_{31}}{2} \\ \frac{\Gamma^0_{21}-\Gamma^0_{12}}{2} & 0 & \frac{c_2-\Gamma^0_{32}}{2} \\ \frac{\Gamma^0_{31}-c_1}{2} & \frac{\Gamma^0_{32}-c_2}{2} & 0 \end{pmatrix}, \tag{3.19}$$

and $\mathrm{tr}(\Gamma_0) = (\Gamma^0_{11}+\Gamma^0_{22}+c_3)$. We have

$$\begin{aligned} W_{\mathrm{curv}}(\Gamma_0) = \mu L_c^2\Big(& b_1\big((\Gamma^0_{11})^2 + \frac{1}{2}(\Gamma^0_{12}+\Gamma^0_{21})^2 + \frac{1}{2}(c_1+\Gamma^0_{31})^2 + (\Gamma^0_{22})^2 + \frac{1}{2}(c_2+\Gamma^0_{32})^2 + c_3^2\big) \\ & + b_2\big(\frac{1}{2}(\Gamma^0_{12}-\Gamma^0_{21})^2 + \frac{1}{2}(c_1-\Gamma^0_{31})^2 + \frac{1}{2}(c_2-\Gamma^0_{32})^2\big) + b_3(\Gamma^0_{11}+\Gamma^0_{22}+c_3)^2\Big)\,. \end{aligned} \tag{3.20}$$

But this is an easy optimisation problem in $\mathbb{R}^3$. Indeed, the stationary points are

$$\begin{aligned} 0 &= \frac{\partial W_{\mathrm{curv}}(\Gamma_0)}{\partial c_1} = b_1(c_1+\Gamma^0_{31}) + b_2(c_1-\Gamma^0_{31}) = (b_1+b_2)c_1 + (b_1-b_2)\Gamma^0_{31} \quad\Rightarrow\quad c_1 = \frac{b_2-b_1}{b_1+b_2}\Gamma^0_{31}\,, \\ 0 &= \frac{\partial W_{\mathrm{curv}}(\Gamma_0)}{\partial c_2} = b_1(c_2+\Gamma^0_{32}) + b_2(c_2-\Gamma^0_{32}) = (b_1+b_2)c_2 + (b_1-b_2)\Gamma^0_{32} \quad\Rightarrow\quad c_2 = \frac{b_2-b_1}{b_1+b_2}\Gamma^0_{32}\,, \\ 0 &= \frac{\partial W_{\mathrm{curv}}(\Gamma_0)}{\partial c_3} = b_1c_3 + b_3(\Gamma^0_{11}+\Gamma^0_{22}+c_3) \quad\Rightarrow\quad c_3 = \frac{-b_3}{b_1+b_3}(\Gamma^0_{11}+\Gamma^0_{22})\,, \end{aligned} \tag{3.21}$$

and the matrix defining the quadratic function in c_1, c_2, c_3 is positive definite, this stationary point is the minimiser, too. By inserting the unknowns inside W_{curv}, we find $W^{\mathrm{hom,\ plate}}_{\mathrm{curv}}$ given by

$$\begin{aligned} W^{\mathrm{hom,\ plate}}_{\mathrm{curv}}(\Gamma) = \mu L_c^2\Big(& b_1\big((\Gamma^0_{11})^2 + (\Gamma^0_{22})^2 + (\frac{-b_3}{b_1+b_3}(\Gamma^0_{11}+\Gamma^0_{22}))^2 + \frac{1}{2}(\Gamma^0_{21}+\Gamma^0_{12})^2 + \frac{1}{2}(\frac{b_2-b_1}{b_1+b_2}\Gamma^0_{31}+\Gamma^0_{31})^2 \\ & + \frac{1}{2}(\frac{b_2-b_1}{b_1+b_2}\Gamma^0_{32}+\Gamma^0_{32})^2\big) + b_2\big(\frac{1}{2}(\Gamma^0_{12}-\Gamma^0_{21})^2 + \frac{1}{2}(\frac{b_2-b_1}{b_1+b_2}\Gamma^0_{31}-\Gamma^0_{31})^2 \\ & + \frac{1}{2}(\frac{b_2-b_1}{b_1+b_2}\Gamma^0_{32}-\Gamma^0_{32})^2\big) + b_3\big((\Gamma^0_{11}+\Gamma^0_{22}) - \frac{b_3}{b_1+b_3}(\Gamma^0_{11}+\Gamma^0_{22})\big)^2\Big) \\ = \mu L_c^2\Big(& b_1\big((\Gamma^0_{11})^2 + (\Gamma^0_{22})^2 + \frac{b_3^2}{(b_1+b_3)^2}(\Gamma^0_{11}+\Gamma^0_{22})^2 + \frac{1}{2}(\Gamma^0_{21}+\Gamma^0_{12})^2 + 2\frac{b_2^2}{(b_1+b_2)^2}(\Gamma^0_{31})^2 \\ & + 2\frac{b_2^2}{(b_1+b_2)^2}(\Gamma^0_{32})^2\big) + b_2\big(\frac{1}{2}(\Gamma^0_{12}-\Gamma^0_{21})^2 + 2\frac{b_1^2}{(b_1+b_2)^2}(\Gamma^0)^2_{31} + 2\frac{b_1^2}{(b_1+b_2)^2}(\Gamma^0_{32})^2\big) \\ & + b_3\frac{b_1^2}{(b_1+b_3)^2}(\Gamma^0_{11}+\Gamma^0_{22})^2\Big) \\ = \mu L_c^2\Big(& b_1\big((\Gamma^0_{11})^2 + (\Gamma^0_{22})^2\big) + \frac{b_1b_3}{(b_1+b_3)}(\Gamma^0_{11}+\Gamma^0_{22})^2 + \frac{b_1}{2}(\Gamma^0_{21}+\Gamma^0_{12})^2 \\ & + 2\frac{b_1b_2}{(b_1+b_2)}(\Gamma^0_{31})^2 + 2\frac{b_1b_2}{(b_1+b_2)}(\Gamma^0_{32})^2 + \frac{b_2}{2}(\Gamma^0_{21}-\Gamma^0_{12})^2\Big) \\ = \mu L_c^2\Big(& b_1\|\mathrm{sym}\Gamma_\square\|^2 + b_2\|\mathrm{skew}\Gamma_\square\|^2 + \frac{b_1b_3}{(b_1+b_3)}\mathrm{tr}(\Gamma_\square)^2 + \frac{2b_1b_2}{(b_1+b_2)}\|\begin{pmatrix}\Gamma^0_{31}\\ \Gamma^0_{32}\end{pmatrix}\|^2\Big)\,, \end{aligned} \tag{3.22}$$

where $\Gamma_\square := \begin{pmatrix}\Gamma^0_{11} & \Gamma^0_{12} \\ \Gamma^0_{21} & \Gamma^0_{22}\end{pmatrix}$.

Therefore, the homogenised curvature energy for the flat Cosserat shell model is

$$\begin{aligned} W_{\rm curv}^{\rm hom,\ plate}(\Gamma) &= \mu L_c^2\Big(b_1\|{\rm sym}\Gamma_\square\|^2 + b_2\|{\rm skew}\Gamma_\square\|^2 + \frac{b_1 b_3}{(b_1+b_3)}{\rm tr}(\Gamma_\square)^2 + \frac{2b_1b_2}{(b_1+b_2)}\|\begin{pmatrix}\Gamma_{31}^0\\ \Gamma_{32}^0\end{pmatrix}\|^2\Big)\\ &= \mu L_c^2\Big(b_1\|{\rm sym}[\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\parallel}\|^2 + b_2\|{\rm skew}[\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\parallel}\|^2 + \frac{b_1 b_3}{(b_1+b_3)}{\rm tr}([\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\parallel})^2 + \frac{2\,b_1b_2}{b_1+b_2}\|[\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\perp}\|\Big) \end{aligned} \tag{3.23}$$

with the orthogonal decomposition in the tangential plane and in the normal direction

$$\mathcal{K}^{\rm plate}_{\overline{R}_0} = [\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\parallel} + [\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\perp}, \qquad [\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\parallel} := e_3\otimes e_3\,\mathcal{K}^{\rm plate}_{\overline{R}_0}, \qquad [\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\perp} := (\mathbb{1}_3 - e_3\otimes e_3)\,\mathcal{K}^{\rm plate}_{\overline{R}_0}\,.$$

□

Since we have now the explicit form of both homogenised energies (membrane and curvature), we are ready to indicate the exact form of the Γ-limit of the sequence of functionals $\mathcal{J}_{h_j}\colon X\to\overline{\mathbb{R}}$ and to provide the following theorem, see [40]

Theorem 3.2. *Assume the boundary data satisfy the conditions*

$$\varphi_d^\natural = \varphi_d\big|_{\partial\Omega_1} \text{(in the sense of traces) for } \varphi_d\in {\rm H}^1(\Omega_1;\mathbb{R}^3), \qquad \Gamma_1\subset\partial\Omega_1 \tag{3.24}$$

and let the constitutive parameters satisfy

$$\mu>0, \qquad \kappa>0, \qquad \mu_{\rm c}>0, \qquad a_1>0, \qquad a_2>0, \qquad a_3>0\,. \tag{3.25}$$

Then, for any sequence $(\varphi^\natural_{h_j},\overline{R}^\natural_{h_j})\in X$ such that $(\varphi^\natural_{h_j},\overline{R}^\natural_{h_j})\to(\varphi_0,\overline{R}_0)$ as $h_j\to 0$, the sequence of functionals $\mathcal{I}_{h_j}\colon X\to\overline{\mathbb{R}}$ from (3.8) Γ-converges to the limit energy functional $\mathcal{I}_0\colon X\to\overline{\mathbb{R}}$ defined by

$$\mathcal{I}_0(m,\overline{R}_0) = \begin{cases} \displaystyle\int_\omega [W_{\rm mp}^{\rm hom,plate}(\mathcal{E}^{\rm plate}_{m,\overline{R}_0}) + \widetilde{W}_{\rm curv}^{\rm hom,plate}(\mathcal{K}^{\rm plate}_{\overline{R}_0})]\,d\omega & \text{if}\quad (m,\overline{R}_0)\in\mathcal{S}'_\omega,\\ +\infty & \text{else in } X, \end{cases} \tag{3.26}$$

where

$$\begin{gathered} m(x_1,x_2) := \varphi_0(x_1,x_2) = \lim_{h_j\to0}\varphi^\natural_{h_j}(x_1,x_2,\frac{1}{h_j}x_3), \qquad \overline{Q}_{e,0}(x_1,x_2) = \lim_{h_j\to0}\overline{R}^\natural_{h_j}(x_1,x_2,\frac{1}{h_j}x_3),\\ \mathcal{E}^{\rm plate}_{m,\overline{R}_0} = \overline{R}_0^T({\rm D}m|0) - \mathbb{1}_2^\flat\,, \qquad \mathcal{K}^{\rm plate}_{\overline{R}_0} = \Big({\rm axl}(\overline{R}_0^T\,\partial_{x_1}\overline{R}_0)\,|\,{\rm axl}(\overline{R}_0^T\,\partial_{x_2}\overline{R}_0)\,|0\Big)\notin{\rm Sym}(3)\,, \end{gathered}$$

and

$$\begin{aligned} W_{\rm mp}^{\rm hom,plate}(\mathcal{E}^{\rm plate}_{m,\overline{R}_0}) &= \mu\,\|{\rm sym}\,[\mathcal{E}^{\rm plate}_{m,\overline{R}_0}]^{\parallel}\|^2 + \mu_{\rm c}\,\|{\rm skew}\,[\mathcal{E}^{\rm plate}_{m,\overline{R}_0}]^{\parallel}\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[{\rm tr}([\mathcal{E}^{\rm plate}_{m,\overline{R}_0}]^{\parallel})\big]^2 + \frac{2\,\mu\,\mu_{\rm c}}{\mu_c+\mu}\|[\mathcal{E}^{\rm plate}_{m,\overline{R}_0}]^T n_0\|^2\\ &= W_{\rm shell}\big([\mathcal{E}^{\rm plate}_{m,\overline{R}_0}]^{\parallel}\big) + \frac{2\,\mu\,\mu_{\rm c}}{\mu_c+\mu}\|[\mathcal{E}^{\rm plate}_{m,\overline{R}_0}]^{\perp}\|^2,\\ \widetilde{W}_{\rm curv}^{\rm hom,plate}(\mathcal{K}^{\rm plate}_{\overline{R}_0}) &= \inf_{A\in\mathfrak{so}(3)}\widetilde{W}_{\rm curv}\Big({\rm axl}(\overline{R}_0^T\,\partial_{\eta_1}\overline{R}_0)\,|\,{\rm axl}(\overline{R}_0^T\,\partial_{\eta_2}\overline{R}_0)\,|\,{\rm axl}(A)\Big)[({\rm D}_x\Theta)^\natural(0)]^{-1}\\ &= \mu L_c^2\Big(b_1\|{\rm sym}[\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\parallel}\|^2 + b_2\|{\rm skew}[\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\parallel}\|^2 + \frac{b_1 b_3}{(b_1+b_3)}{\rm tr}([\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\parallel})^2 + \frac{2\,b_1b_2}{b_1+b_2}\|[\mathcal{K}^{\rm plate}_{\overline{R}_0}]^{\perp}\|\Big)\,. \end{aligned} \tag{3.27}$$

4. Homogenised curvature energy for the curved Cosserat shell model via Γ-convergence

In this section, we consider the case of a curved Cosserat shell model and we give the explicit form and the detailed calculation of the homogenised curvature energy. In comparison with the flat Cosserat shell model, the calculations are more complicated. Hence, let us consider an elastic material which in its reference configuration fills the three-dimensional *shell-like thin* domain $\Omega_\xi \subset \mathbb{R}^3$, i.e. we assume that there exists a C^1-diffeomorphism $\Theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $\Theta(x_1,x_2,x_3) := (\xi_1,\xi_2,\xi_3)$ such that $\Theta(\Omega_h) = \Omega_\xi$ and $\omega_\xi = \Theta(\omega\times\{0\})$, where $\Omega_h \subset \mathbb{R}^3$ for $\Omega_h = \omega \times \left[-\frac{h}{2},\frac{h}{2}\right]$, with $\omega \subset \mathbb{R}^2$ a bounded domain with Lipschitz boundary $\partial\omega$. The scalar $0<h\ll 1$ is called *thickness* of the shell, while the domain Ω_h is called *fictitious Cartesian configuration* of the body. In fact, in this paper, we consider the following diffeomorphism $\Theta\colon \mathbb{R}^3 \to \mathbb{R}^3$ which describes the curved surface of the shell

$$\Theta(x_1,x_2,x_3) = y_0(x_1,x_2) + x_3\, n_0(x_1,x_2)\,, \tag{4.1}$$

where $y_0\colon \omega \to \mathbb{R}^3$ is a $C^2(\omega)$-function and $n_0 = \frac{\partial_{x_1} y_0 \times \partial_{x_2} y_0}{\|\partial_{x_1} y_0 \times \partial_{x_2} y_0\|}$ is the unit normal vector on ω_ξ. Remark that

$$\mathrm{D}_x\Theta(x_3) \;=\; (\mathrm{D}y_0|n_0) + x_3(\mathrm{D}n_0|0)\ \ \forall\, x_3 \in \left(-\frac{h}{2},\frac{h}{2}\right),\quad \mathrm{D}_x\Theta(0) \;=\; (\mathrm{D}y_0|\,n_0),\quad [\mathrm{D}_x\Theta(0)]^{-T}\,e_3 \;=\; n_0, \tag{4.2}$$

and $\det \mathrm{D}_x\Theta(0) = \det(\mathrm{D}y_0|n_0) = \sqrt{\det[(\mathrm{D}y_0)^T\mathrm{D}y_0]}$ represents the surface element. We also have the polar decomposition $\mathrm{D}_x\Theta(0) = Q_0\,U_0$, where

$$Q_0 = \mathrm{polar}(\mathrm{D}_x\Theta(0)) = \mathrm{polar}([\mathrm{D}_x\Theta(0)]^{-T}) \in \mathrm{SO}(3) \quad \text{and} \quad U_0 \in \mathrm{Sym}^+(3)\,. \tag{4.3}$$

The first step in our shell model is to transform the problem to a variational problem defined on the fictitious flat configuration $\Omega_h = \omega \times \left[-\frac{h}{2},\frac{h}{2}\right]$. The next step, in order to apply the Γ-convergence technique, is to transform the minimisation problem onto the *fixed domain* Ω_1, which is independent of the thickness h. These two steps were done in [40], the three-dimensional problem (2.1) (corresponding to the Cosserat curvature tensor α) being equivalent to the following minimisation problem on Ω_1

$$\begin{aligned}
I_h^\natural(\varphi^\natural, \mathrm{D}_\eta^h\varphi^\natural, \overline{Q}_{e,h}^\natural, \Gamma_{e,h}^\natural) &= \int_{\Omega_1} \left(W_{\mathrm{mp}}(U_{e,h}^\natural) + \widetilde{W}_{\mathrm{curv}}(\Gamma_{e,h}^\natural)\right) \det(\mathrm{D}_\eta\zeta(\eta))\det((\mathrm{D}_x\Theta)^\natural(\eta_3))\, dV_\eta \\
&= \int_{\Omega_1} h\,\left[\left(W_{\mathrm{mp}}(U_{e,h}^\natural) + \widetilde{W}_{\mathrm{curv}}(\Gamma_{e,h}^\natural)\right)\det((\mathrm{D}_x\Theta)^\natural(\eta_3))\right] dV_\eta \mapsto \min \ \text{w.r.t}\ (\varphi^\natural, \overline{Q}_{e,h}^\natural)\,,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
W_{\mathrm{mp}}(U_{e,h}^\natural) &= \ \mu\,\|\mathrm{sym}(U_{e,h}^\natural - \mathbb{1}_3)\|^2 + \mu_c\,\|\mathrm{skew}(U_{e,h}^\natural - \mathbb{1}_3)\|^2 + \frac{\lambda}{2}[\mathrm{tr}(\mathrm{sym}(U_{e,h}^\natural - \mathbb{1}_3))]^2\,,\\
\widetilde{W}_{\mathrm{curv}}(\Gamma_{e,h}^\natural) &= \ \mu\,L_c^2\left(b_1\,\|\,\mathrm{sym}\,\Gamma_{e,h}^\natural\|^2 + b_2\,\|\mathrm{skew}\,\Gamma_{e,h}^\natural\|^2 + \,b_3\,[\mathrm{tr}(\Gamma_{e,h}^\natural)]^2\right)\,,\\
U_{e,h}^\natural &= \overline{Q}_{e,h}^{\natural,T} F_h^\natural[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1} = \overline{Q}_{e,h}^{\natural,T}\mathrm{D}_\eta^h\varphi^\natural(\eta)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\,,\\
\Gamma_{e,h}^\natural &= \left(\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_1}\overline{Q}_{e,h}^\natural)\,|\,\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_2}\overline{Q}_{e,h}^\natural)\,|\,\frac{1}{h}\mathrm{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_3}\overline{Q}_{e,h}^\natural)\right)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1},
\end{aligned} \tag{4.5}$$

with $(\mathrm{D}_x\Theta)^\natural(\eta_3)$ the nonlinear scaling (see (3.2)) of $\mathrm{D}_x\Theta$, $F_h^\natural = \mathrm{D}_\eta^h\varphi^\natural$ the nonlinear scaling of the gradient of the mapping $\varphi\colon \Omega_h \to \Omega_c\,,\ \varphi(x_1,x_2,x_3) = \varphi_\xi(\Theta(x_1,x_2,x_3))\,,\ \overline{Q}_{e,h}^\natural$ the nonlinear scaling of the *elastic microrotation* $\overline{Q}_e\colon \Omega_h \to \mathrm{SO}(3)$ defined by $\overline{Q}_e(x_1,x_2,x_3) := \overline{R}_\xi(\Theta(x_1,x_2,x_3))$. Since for $\eta_3 = 0$ the values of $\mathrm{D}_x\Theta$, Q_0, U_0 expressed in terms of $(\eta_1,\eta_2,0)$ and $(x_1,x_2,0)$ coincide, we will omit the sign $\cdot^\natural$ and we will understand from the context the variables into discussion, i.e.

$$\begin{aligned}(\mathrm{D}_x\Theta)(0) &:= (\mathrm{D}y_0\,|n_0) = (\mathrm{D}_x\Theta)^\natural(\eta_1,\eta_2,0) \equiv (\mathrm{D}_x\Theta)(x_1,x_2,0),\\ Q_0(0) &:= Q_0^\natural(\eta_1,\eta_2,0) \equiv Q_0(x_1,x_2,0), \qquad\qquad U_0(0) := U_0^\natural(\eta_1,\eta_2,0) \equiv U_0(x_1,x_2,0).\end{aligned} \tag{4.6}$$

In order to construct the Γ-limit of the rescaled energies

$$\mathcal{I}_h^\natural(\varphi^\natural, \mathrm{D}_\eta^h\varphi^\natural, \overline{R}_h^\natural, \Gamma_h^\natural) = \begin{cases} \dfrac{1}{h}\, I_h^\natural(\varphi^\natural, \mathrm{D}_\eta^h\varphi^\natural, \overline{Q}_{e,h}^\natural, \Gamma_{e,h}^\natural) & \text{if } (\varphi^\natural, \overline{Q}_{e,h}^\natural) \in \mathcal{S}',\\ +\infty & \text{else in } X,\end{cases} \tag{4.7}$$

for curved Cosserat shell model, we have to solve the following **four (not only two as for flat Cosserat shell models)** auxiliary optimisation problems.

O1: For each $\varphi^\natural : \Omega_1 \to \mathbb{R}^3$ and $\overline{Q}_{e,h}^\natural : \Omega_1 \to \mathrm{SO}(3)$ we determine a vector $d^* \in \mathbb{R}^3$ through

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{hom},\natural}(\mathcal{E}_{\varphi^\natural,\overline{Q}_{e,h}^\natural}) &:= W_{\mathrm{mp}}\Big(\overline{Q}_{e,h}^{\natural,T}(\mathrm{D}_{(\eta_1,\eta_2)}\varphi^\natural|d^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\Big)\\ &= \inf_{c\in\mathbb{R}^3} W_{\mathrm{mp}}\Big(\overline{Q}_{e,h}^{\natural,T}(\mathrm{D}_{(\eta_1,\eta_2)}\varphi^\natural|c)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\Big), \end{aligned} \tag{4.8}$$

where $\mathcal{E}_{\varphi^\natural,\overline{Q}_{e,h}^\natural} := (\overline{Q}_e^{\natural,T}\mathrm{D}_{(\eta_1,\eta_2)}\varphi^\natural - (\mathrm{D}y_0)^\natural|0)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}$ represents the non-fully[3] dimensional reduced elastic shell strain tensor.

O2: For each pair $(m, \overline{Q}_{e,0})$, where $m:\omega\to\mathbb{R}^3$, $\overline{Q}_{e,0}:\omega\to\mathrm{SO}(3)$ we determine the vector $\widetilde{d}^* \in \mathbb{R}^3$ through

$$\begin{aligned} W_{\mathrm{mp}}^{\mathrm{hom}}(\mathcal{E}_{m,\overline{Q}_{e,0}}) := W_{\mathrm{mp}}\Big(\overline{Q}_{e,0}^{T}(\mathrm{D}m|\widetilde{d}^*)[(\mathrm{D}_x\Theta)(0)]^{-1}\Big) &= \inf_{\widetilde{c}\in\mathbb{R}^3} W_{\mathrm{mp}}\Big(\overline{Q}_{e,0}^{T}(\mathrm{D}m|\widetilde{c})[(\mathrm{D}_x\Theta)(0)]^{-1}\Big)\\ &= \inf_{\widetilde{c}\in\mathbb{R}^3} W_{\mathrm{mp}}\Big(\mathcal{E}_{m,\overline{Q}_{e,0}} - (0|0|\widetilde{c})[(\mathrm{D}_x\Theta)(0)]^{-1}\Big), \end{aligned} \tag{4.9}$$

where $\mathcal{E}_{m,\overline{Q}_{e,0}} := (\overline{Q}_{e,0}^{T}\mathrm{D}m - \mathrm{D}y_0|0)[\mathrm{D}_x\Theta(0)]^{-1}$ represents the *elastic shell strain tensor.*

O3: For each $\overline{Q}_{e,h}^\natural : \Omega_1 \to \mathrm{SO}(3)$, we determine the skew-symmetric matrix $A^* \in \mathfrak{so}\,3$, i.e. its axial vector $\operatorname{axl} A^* \in \mathbb{R}^3$ through

$$\begin{aligned} \widetilde{W}_{\mathrm{curv}}^{\mathrm{hom},\natural}(\mathcal{K}_{\overline{Q}_{e,h}^\natural}) &:= \widetilde{W}_{\mathrm{curv}}\Big(\big(\operatorname{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_1}\overline{Q}_{e,h}^\natural)\,|\,\operatorname{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_2}\overline{Q}_{e,h}^\natural)\,|\,\operatorname{axl}(A^*)\big)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\Big)\\ &= \inf_{A\in\mathfrak{so}(3)} \widetilde{W}_{\mathrm{curv}}\Big(\big(\operatorname{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_1}\overline{Q}_{e,h}^\natural)\,|\,\operatorname{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_2}\overline{Q}_{e,h}^\natural)\,|\,\operatorname{axl}(A)\big)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\Big), \end{aligned} \tag{4.10}$$

where $\mathcal{K}_{\overline{Q}_{e,h}^\natural} := \Big(\operatorname{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_1}\overline{Q}_{e,h}^\natural)\,|\,\operatorname{axl}(\overline{Q}_{e,h}^{\natural,T}\,\partial_{\eta_2}\overline{Q}_{e,h}^\natural)\,|0\Big)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}$ represents a not fully reduced elastic shell bending curvature tensor, in the sense that it still depends on η_3 and h, since $\overline{Q}_{e,h}^\natural = \overline{Q}_{e,h}^\natural(\eta_1,\eta_2,\eta_3)$. Therefore, $\widetilde{W}_{\mathrm{curv}}^{\mathrm{hom},\natural}(\mathcal{K}_{\overline{Q}_{e,h}^\natural})$ given by the above definitions still depends on η_3 and h.

[3]Here, "non-fully" means that the introduced quantities still depend on η_3 and h, because the elements $\mathrm{D}_{(\eta_1,\eta_2)}\varphi^\natural$ still depend on η_3 and $\overline{Q}^{\natural,T}$ depends on h.

O4: For each $\overline{Q}_{e,0} : \Omega_1 \to \mathrm{SO}(3)$, we determine the skew-symmetric matrix $A^* \in \mathfrak{so}(3)$, i.e. its axial vector $\operatorname{axl} A^* \in \mathbb{R}^3$, though

$$\begin{aligned}\widetilde{W}^{\mathrm{hom}}_{\mathrm{curv}}(\mathcal{K}_{\overline{Q}_{e,0}}) :&= \widetilde{W}^{*}_{\mathrm{curv}}\Big(\big(\operatorname{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_1}\overline{Q}_{e,0})\,|\,\operatorname{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_2}\overline{Q}_{e,0})\,|\,\operatorname{axl}(A^*)\big)[(\mathrm{D}_x\Theta)^{\natural}(0)]^{-1}\Big)\\ &= \inf_{A\in\mathfrak{so}(3)} \widetilde{W}_{\mathrm{curv}}\Big(\big(\operatorname{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_1}\overline{Q}_{e,0})\,|\,\operatorname{axl}(\overline{Q}_{e,0}^T\,\partial_{\eta_2}\overline{Q}_{e,0})\,|\,\operatorname{axl}(A)\big)[(\mathrm{D}_x\Theta)^{\natural}(0)]^{-1}\Big), \quad (4.11)\end{aligned}$$

where $\mathcal{K}_{\overline{Q}_{e,0}} := \Big(\operatorname{axl}(\overline{Q}_{e,0}^T\,\partial_{x_1}\overline{Q}_{e,0})\,|\,\operatorname{axl}(\overline{Q}_{e,0}^T\,\partial_{x_2}\overline{Q}_{e,0})\,|0\Big)[\mathrm{D}_x\Theta(0)]^{-1} \notin \mathrm{Sym}(3)$ represents the *the elastic shell bending curvature tensor.*

Let us remark that having the solutions of the optimisation problems O1 and O3, the solutions for the optimisation problems O2 and O4, respectively, follow immediately. However, we cannot skip the solution of the optimisation problems O1 and O3 and use only the solutions of the optimisation problems O2 and O4, since the knowledge of $W^{\mathrm{hom}}_{\mathrm{mp}}$ and $W^{\mathrm{hom}}_{\mathrm{curv}}$ is important in the proof of the Γ-convergence result. This is the first major difference between Γ-convergence for curved initial configurations and flat initial configuration.

The solutions of the first two optimisation problems and the complete calculations where given in [40], while the analytical calculations of the last two optimisation problems were left open until now.

For the completeness of the exposition, we recall the following result

Theorem 4.1. [40] *The solution of the optimisation problem O2 is*

$$\widetilde{d}^* = \Big(1 - \frac{\lambda}{2\,\mu\,+\lambda}\langle \mathcal{E}_{m,\overline{Q}_{e,0}}, \mathbb{1}_3\rangle\Big)\overline{Q}_{e,0}n_0 + \frac{\mu_c-\mu}{\mu_c\,+\mu}\,\overline{Q}_{e,0}\mathcal{E}^T_{m,\overline{Q}_{e,0}}n_0\,, \tag{4.12}$$

and

$$\begin{aligned}W^{\mathrm{hom}}_{\mathrm{mp}}(\mathcal{E}_{m,\overline{Q}_{e,0}}) &= \mu\,\|\mathrm{sym}\,\mathcal{E}^{\|}_{m,\overline{Q}_{e,0}}\|^2 + \mu_c\,\|\mathrm{skew}\,\mathcal{E}^{\|}_{m,\overline{Q}_{e,0}}\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}^{\|}_{m,\overline{Q}_{e,0}})\big]^2 + \frac{2\,\mu\,\mu_c}{\mu_c\,+\mu}\|\mathcal{E}^T_{m,\overline{Q}_{e,0}}n_0\|^2\\ &= W_{\mathrm{shell}}\big(\mathcal{E}^{\|}_{m,\overline{Q}_{e,0}}\big) + \frac{2\,\mu\,\mu_c}{\mu_c\,+\mu}\|\mathcal{E}^{\perp}_{m,\overline{Q}_{e,0}}\|^2, \end{aligned}\tag{4.13}$$

where $W_{\mathrm{shell}}\big(\mathcal{E}^{\|}_{m,\overline{Q}_{e,0}}\big) = \mu\,\|\mathrm{sym}\,\mathcal{E}^{\|}_{m,\overline{Q}_{e,0}}\|^2 + \mu_c\,\|\mathrm{skew}\,\mathcal{E}^{\|}_{m,\overline{Q}_{e,0}}\|^2 + \dfrac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}^{\|}_{m,\overline{Q}_{e,0}})\big]^2$ *with the orthogonal decomposition in the tangential plane and in the normal direction*

$$\mathcal{E}_{m,\overline{Q}_{e,0}} = \mathcal{E}^{\|}_{m,\overline{Q}_{e,0}} + \mathcal{E}^{\perp}_{m,\overline{Q}_{e,0}}, \qquad \mathcal{E}^{\|}_{m,\overline{Q}_{e,0}} := \mathrm{A}_{y_0}\,\mathcal{E}_{m,\overline{Q}_{e,0}}, \qquad \mathcal{E}^{\perp}_{m,\overline{Q}_{e,0}} := (\mathbb{1}_3 - \mathrm{A}_{y_0})\,\mathcal{E}_{m,\overline{Q}_{e,0}}, \tag{4.14}$$

and $\mathrm{A}_{y_0} := (\mathrm{D}y_0|0)\;[\mathrm{D}\Theta_x(0)]^{-1} \in \mathbb{R}^{3\times 3}$.

In the remainder of this section, we provide the explicit solutions of the optimisation problems O3 and O4. We remark that, while in the case of flat initial configuration the solution of O4 is very easy to be found, in the case of curved initial configuration the calculations are more difficult. Beside this, for curved initial configurations there is a need to solve the optimisation problem O3, too. Notice that for flat initial configuration the optimisation problems O3 and O4 coincide.

4.1. The calculation of the homogenised curvature energy

We have the following isotropic curvature energy formula for a curved configuration

$$\widetilde{W}_{\mathrm{curv}}(\Gamma^{\natural}_{e,h}) = \mu L_c^2\Big(b_1\|\mathrm{sym}\,\Gamma^{\natural}_{e,h}\|^2 + b_2\,\|\mathrm{skew}\Gamma^{\natural}_{e,h}\|^2 + b_3\mathrm{tr}(\Gamma^{\natural}_{e,h})^2\Big)\,. \tag{4.15}$$

Theorem 4.2. *The solution of the optimisation problem O3 given by* (4.10) *is*

$$c^* = \frac{(b_2 - b_1)}{b_1 + b_2}\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\, n_0 - \frac{2b_3}{2(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})n_0 \tag{4.16}$$

and the corresponding homogenised curvature energy is

$$W^{hom}_{curv}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}) = \mu L_c^2\Big(b_1\|\mathrm{sym}\mathcal{K}^{\parallel}_{\overline{Q}^\natural_{e,h}}\|^2 + b_2\|\mathrm{skew}\mathcal{K}^{\parallel}_{\overline{Q}^\natural_{e,h}}\|^2 + \frac{b_1 b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}^{\parallel}_{\overline{Q}^\natural_{e,h}})^2 + \frac{2b_1b_2}{b_1+b_2}\|\mathcal{K}^{\perp}_{\overline{Q}^\natural_{e,h}}\|\Big),$$

where $\mathcal{K}^{\parallel}_{\overline{Q}^\natural_{e,h}}$ *and* $\mathcal{K}^{\perp}_{\overline{Q}^\natural_{e,h}}$ *represent the orthogonal decomposition of the a not fully reduced elastic shell bending curvature tensor* $\mathcal{K}_{\overline{Q}^\natural_{e,h}}$ *in the tangential plane and in the normal direction, respectively.*

Proof. We need to find

$$\widetilde{W}^{\mathrm{hom},\natural}_{\mathrm{curv}}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}) = \widetilde{W}_{\mathrm{curv}}(\mathcal{K}_{\overline{Q}^\natural_{e,h}} + (0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}) = \inf_{c\in\mathbb{R}^3}\widetilde{W}_{\mathrm{curv}}(\underbrace{\mathcal{K}_{\overline{Q}^\natural_{e,h}} + (0|0|c)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}}_{=:\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}}). \tag{4.17}$$

The Euler–Lagrange equations appear from variations with respect to arbitrary increments $\delta c \in \mathbb{R}^3$.

$$\begin{aligned}\langle DW_{\mathrm{curv}}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}), (0|0|\delta c)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\rangle = 0 \quad &\Leftrightarrow \quad \langle[D\widetilde{W}_{\mathrm{curv}}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}})]\,[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-T}, e_3\otimes\delta c\rangle = 0\\ &\Leftrightarrow \quad \langle[D\widetilde{W}_{\mathrm{curv}}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}})]\,[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-T}\, e_3, \delta c\rangle = 0\\ &\Leftrightarrow \quad \langle[D\widetilde{W}_{\mathrm{curv}}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}})]\, n_0, \delta c\rangle = 0 \quad \forall \delta c\in\mathbb{R}^3.\end{aligned} \tag{4.18}$$

Therefore, we deduce that if c^* is a minimum then

$$[D\widetilde{W}_{\mathrm{curv}}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}})]\, n_0 = 0 \quad \Leftrightarrow \quad \Big(2a_1\mathrm{sym}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}) + 2\,a_2\mathrm{skew}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}) + 2a_3\mathrm{tr}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}})\Big)n_0 = 0\,. \tag{4.19}$$

Since $\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}} = \mathcal{K}_{\overline{Q}^\natural_{e,h}} + (0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}$, we have

$$\begin{aligned}2\mathrm{sym}\big(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}\big)n_0 &= 2\Big(\mathrm{sym}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}) + \mathrm{sym}((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})\Big)n_0\\ &= \Big(\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h})\,|\,\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h})\,|0\Big)\underbrace{[(\mathrm{D}_x\Theta)^\natural]^{-1}(\eta_3)\,n_0}_{=e_3} + \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\,n_0\\ &\quad + (0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}n_0 + ((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})^T n_0\\ &= \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\,n_0 + c^* + ((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})^T n_0\,.\end{aligned} \tag{4.20}$$

Similar calculations show that

$$\begin{aligned}2\mathrm{skew}\big(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}\big)n_0 &= 2\Big(\mathrm{skew}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}) + \mathrm{skew}((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})\Big)n_0\\ &= -\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\,n_0 + c^* - ((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})^T n_0\,,\end{aligned} \tag{4.21}$$

while the trace term is calculated to be

$$\begin{aligned}2\,\mathrm{tr}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}})n_0 &= 2\Big(\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}) + \mathrm{tr}((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})\Big)n_0\\ &= 2\,\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})n_0 + 2\langle(0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}, \mathbb{1}_3\rangle_{\mathbb{R}^{3\times3}}\,n_0\\ &= 2\,\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})n_0 + 2\langle c^*, \underbrace{[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-T}e_3}_{=n_0}\rangle n_0 = 2\,\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})n_0 + 2\,c^*\,n_0\otimes n_0\,.\end{aligned} \tag{4.22}$$

By using (4.19), we obtain

$$\begin{aligned} & b_1\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0 + b_1c^* + b_1((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})^Tn_0 - b_2\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0 + b_2c^* \\ & \quad - b_2((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})^Tn_0 + 2b_3\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})n_0 + 2b_3\, c^*\, n_0\otimes n_0 = 0\,. \end{aligned} \tag{4.23}$$

Gathering similar terms gives us

$$\begin{aligned} & (b_1-b_2)\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0 + (b_1+b_2)c^* + (b_1-b_2)((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})^Tn_0 \\ & \quad + 2b_3\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})n_0 + 2b_3\, c^*\, n_0\otimes n_0 = 0\,. \end{aligned} \tag{4.24}$$

We have

$$\begin{aligned} ((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})^Tn_0 &= (c^*\,(0|0|e_3)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})^Tn_0 = (c^*\,n_0)^Tn_0 \\ &= n_0^T{c^*}^Tn_0 = \langle n_0, {c^*}^T\rangle n_0 = n_0\langle n_0, c^*\rangle = n_0\otimes n_0\, c^* = c^*\, n_0\otimes n_0\,, \end{aligned} \tag{4.25}$$

and by using the decomposition [4–6] $\mathbb{1}_3\, c^* = A_{y_0}\, c^* + n_0\otimes n_0\, c^*$, we obtain

$$\begin{aligned} & (b_1-b_2)\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0 + (b_1+b_2)(A_{y_0}\, c^* + n_0\otimes n_0\, c^*) + (b_1-b_2)n_0\otimes n_0\, c^* \\ & \quad + 2b_3\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})n_0 + 2b_3\, n_0\otimes n_0\, c^* = 0\,, \end{aligned} \tag{4.26}$$

and

$$[(b_1+b_2)A_{y_0} + 2(b_1+b_3)n_0\otimes n_0]\, c^* = -(b_1-b_2)\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0 - 2b_3\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\, n_0\,. \tag{4.27}$$

Since A_{y_0} is orthogonal to $n_0\otimes n_0$ and $A^2_{y_0} = A_{y_0}$,

$$\left[\frac{1}{b_1+b_2}A_{y_0} + \frac{1}{2(b_1+b_3)}n_0\otimes n_0\right][(b_1+b_2)A_{y_0} + 2(b_1+b_3)n_0\otimes n_0] = \mathbb{1}_3 \tag{4.28}$$

(see [5]), we have

$$[(b_1+b_2)A_{y_0} + 2(b_1+b_3)n_0\otimes n_0]^{-1} = \frac{1}{b_1+b_2}A_{y_0} + \frac{1}{2(b_1+b_3)}n_0\otimes n_0 \tag{4.29}$$

and we find

$$c^* = (b_2-b_1)\Big[\frac{1}{b_1+b_2}A_{y_0} + \frac{1}{2(b_1+b_3)}n_0\otimes n_0\Big]\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0 - 2b_3\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\Big[\frac{1}{b_1+b_2}A_{y_0} + \frac{1}{2(b_1+b_3)}n_0\otimes n_0\Big]n_0\,.$$

Because

$$\begin{aligned} A_{y_0}\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} &= \mathbb{1}_3\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} - n_0\otimes n_0\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} \\ &= \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} - (0|0|n_0)(0|0|n_0)^T[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-T}\Big(\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h})\,|\,\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h})\,|0\Big)^Tn_0 \\ &= \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} - (0|0|n_0)(\underbrace{[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}(0|0|n_0)}_{(0|0|e_3)})^T\Big(\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h})\,|\,\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h})\,|0\Big)^Tn_0 \\ &= \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} - (0|0|n_0)(0|0|e_3)^T\Big(\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h})\,|\,\mathrm{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h})\,|0\Big)^Tn_0 \\ &= \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} - (0|0|n_0)\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}*&*&*\\ *&*&*\\0&0&0\end{pmatrix}n_0 = \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\,, \end{aligned} \tag{4.30}$$

we obtain the unique minimiser

$$c^* = \frac{(b_2-b_1)}{b_1+b_2}\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\, n_0 - \frac{2b_3}{2(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\, n_0\,. \tag{4.31}$$

Next, we insert the minimiser c^* in (4.31). We have

$$\begin{aligned}
\|\mathrm{sym}\,\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}\|^2 &= \|\mathrm{sym}\big(\mathcal{K}_{\overline{Q}^\natural_{e,h}}\big)\|^2 + \|\mathrm{sym}\big((0|0|c)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\big)\|^2\\
&\quad + 2\left\langle \mathrm{sym}\big(\mathcal{K}_{\overline{Q}^\natural_{e,h}}\big), \mathrm{sym}\big((0|0|c)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\big)\right\rangle\\
&= \|\mathrm{sym}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\|^2\\
&\quad + \|\mathrm{sym}\Big(\frac{b_2-b_1}{b_1+b_2}\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}(0|0|n_0)[(\mathrm{D}_x\Theta)^\natural]^{-1} - \frac{b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})(0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\Big)\|^2\\
&\quad + 2\Big\langle \mathrm{sym}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}), \mathrm{sym}\Big(\frac{b_2-b_1}{b_1+b_2}\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}(0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\\
&\qquad - \frac{b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})(0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\Big)\Big\rangle,
\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}
&\|\mathrm{sym}\Big(\frac{b_2-b_1}{b_1+b_2}\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}(0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1} - \frac{b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})(0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}\Big)\|^2\\
&= \frac{(b_2-b_1)^2}{(b_1+b_2)^2}\|\mathrm{sym}(\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0)\|^2 + \frac{b_3^2}{(b_1+b_3)^2}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2\|n_0\otimes n_0\|^2\\
&\quad - 2\,\frac{b_2-b_1}{b_1+b_2}\,\frac{b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\langle \mathrm{sym}(\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0), n_0\otimes n_0\rangle\\
&= \frac{(b_2-b_1)^2}{(b_1+b_2)^2}\left\langle \mathrm{sym}(\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0), \mathrm{sym}(\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0)\right\rangle + \frac{b_3^2}{(b_1+b_3)^2}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2\\
&\quad - \frac{b_2-b_1}{b_1+b_2}\,\frac{b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0, n_0\otimes n_0\rangle\\
&\quad - \frac{b_2-b_1}{b_1+b_2}\,\frac{b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\langle n_0\otimes n_0\,\mathcal{K}_{\overline{Q}^\natural_{e,h}}, n_0\otimes n_0\rangle\\
&= \frac{(b_2-b_1)^2}{4(b_1+b_2)^2}\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0, \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0\rangle + \frac{(b_2-b_1)^2}{4(b_1+b_2)^2}\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0, n_0\otimes n_0\,\mathcal{K}_{\overline{Q}^\natural_{e,h}}\rangle\\
&\quad + \frac{(b_2-b_1)^2}{4(b_1+b_2)^2}\langle n_0\otimes n_0\,\mathcal{K}_{\overline{Q}^\natural_{e,h}}, \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\otimes n_0\rangle + \frac{(b_2-b_1)^2}{4(b_1+b_2)^2}\langle n_0\otimes n_0\,\mathcal{K}_{\overline{Q}^\natural_{e,h}}, n_0\otimes n_0\,\mathcal{K}_{\overline{Q}^\natural_{e,h}}\rangle\\
&\quad + \frac{b_3^2}{(b_1+b_3)^2}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2 = \frac{(b_2-b_1)^2}{2(b_1+b_2)^2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}n_0\|^2 + \frac{b_3^2}{(b_1+b_3)^2}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2.
\end{aligned} \tag{4.33}$$

Note that

$$
\begin{aligned}
&\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}, n_0\otimes n_0\rangle = \langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}, n_0\otimes n_0\rangle\\
&= \langle\Big(\operatorname{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h})\,|\,\operatorname{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h})\,|0\Big)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1}, (0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(0)]^{-1}\rangle\\
&= \left\langle \Big(\operatorname{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h})\,|\,\operatorname{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h})\,|0\Big), (0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(0)]^{-1}[(\mathrm{D}_x\Theta)^\natural(0)]^{-T}\begin{pmatrix}\mathbb{1}_2 - x_3 L_{y_0} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}^{-T}\right\rangle\\
&= \left\langle (0|0|n_0)^T\Big(\operatorname{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_1}\overline{Q}^\natural_{e,h})\,|\,\operatorname{axl}(\overline{Q}^{\natural,T}_{e,h}\,\partial_{\eta_2}\overline{Q}^\natural_{e,h})\,|0\Big), \begin{pmatrix}\mathrm{I}_{y_0}^{-1} & 0\\ & 0\\ 0\quad 0 & 1\end{pmatrix}\begin{pmatrix}\mathbb{1}_2 - x_3 L_{y_0} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}^{-T}\right\rangle\\
&= \left\langle \begin{pmatrix}0&0&0\\0&0&0\\ *&*&0\end{pmatrix}, \begin{pmatrix}*&*&0\\ *&*&0\\0&0&1\end{pmatrix}\right\rangle = 0\,,
\end{aligned}\tag{4.34}
$$

where the *Weingarten map (or shape operator)* is defined by $\mathrm{L}_{y_0} = \mathrm{I}_{y_0}^{-1}\mathrm{II}_{y_0} \in \mathbb{R}^{2\times 2}$, where $\mathrm{I}_{y_0} := [\mathrm{D}y_0]^T\,\mathrm{D}y_0 \in \mathbb{R}^{2\times 2}$ and $\mathrm{II}_{y_0} := -[\mathrm{D}y_0]^T\,\mathrm{D}n_0 \in \mathbb{R}^{2\times 2}$ are the matrix representations of the *first fundamental form (metric)* and the *second fundamental form* of the surface, respectively. We also observe that

$$
\begin{aligned}
n_0\otimes n_0[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-T} &= (0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(0)]^{-1}[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-T}\\
&= (0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(0)]^{-1}[(\mathrm{D}_x\Theta)^\natural(0)]^{-T}\begin{pmatrix}\mathbb{1}_2 - x_3 L_{y_0} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}^{-T}\\
&= (0|0|n_0)\begin{pmatrix}\mathrm{I}_{y_0}^{-1} & 0\\ & 0\\ 0\quad 0 & 1\end{pmatrix}\begin{pmatrix}\mathbb{1}_2 - x_3 L_{y_0} & 0\\ & 0\\ 0 \quad 0 & 1\end{pmatrix}^{-T} = (0|0|n_0)\begin{pmatrix}*&*&0\\ *&*&0\\0&0&1\end{pmatrix} = (0|0|n_0)\,.
\end{aligned}\tag{4.35}
$$

For every vector $\widehat{u}, v\in\mathbb{R}^3$ we have

$$
\langle \widehat{u}\otimes n_0, v\otimes n_0\rangle = \langle (v\otimes n_0)^T\widehat{u}\otimes n_0, \mathbb{1}\rangle = \langle (n_0\otimes v)\widehat{u}\otimes n_0, \mathbb{1}\rangle = \langle n_0\otimes n_0\langle v,\widehat{u}\rangle, \mathbb{1}\rangle = \langle v,\widehat{u}\rangle\cdot\underbrace{\langle n_0, n_0\rangle}_{=1} = \langle v,\widehat{u}\rangle\,,
$$

and $n_0\otimes n_0 = (0|0|n_0)[(\mathrm{D}_x\Theta)^\natural(0)]^{-1}$, we deduce

$$
\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0, \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0\rangle = \langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0, \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\rangle = \|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2\,. \tag{4.36}
$$

On the other hand,

$$
2\left\langle \operatorname{sym}\mathcal{K}_{\overline{Q}^\natural_{e,h}}, \operatorname{sym}\Big(\frac{b_2-b_1}{b_1+b_2}\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0 - \frac{b_3}{(b_1+b_3)}\operatorname{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})n_0\otimes n_0\Big)\right\rangle = \frac{b_2-b_1}{b_1+b_2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2\,. \tag{4.37}
$$

Therefore,

$$
\|\operatorname{sym}\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}\|^2 = \|\operatorname{sym}\mathcal{K}_{\overline{Q}^\natural_{e,h}}\|^2 + \frac{(b_1-b_2)^2}{2(b_1+b_2)^2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2 + \frac{b_3^2}{(b_1+b_3)^2}\operatorname{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2 + \frac{b_2-b_1}{b_1+b_2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2\,. \tag{4.38}
$$

Now we continue the calculations for the skew-symmetric part,

$$
\begin{aligned}
\|\operatorname{skew}\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}\|^2 =& \|\operatorname{skew}\mathcal{K}_{\overline{Q}^\natural_{e,h}}\|^2 + \|\operatorname{skew}((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})\|^2\\
&+ 2\langle \operatorname{skew}\mathcal{K}_{\overline{Q}^\natural_{e,h}}, \operatorname{skew}((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})\rangle.
\end{aligned}\tag{4.39}
$$

In a similar manner, we calculate the terms separately. Since $n_0 \otimes n_0$ is symmetric, we obtain

$$\begin{aligned}\|\mathrm{skew}((0|0|c)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})\|^2 &= \|\mathrm{skew}(\frac{b_2-b_1}{b_1+b_2}\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\, n_0\otimes n_0 - \frac{b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\, n_0\otimes n_0)\|^2\\ &= \frac{(b_1-b_2)^2}{(b_1+b_2)^2}\|\mathrm{skew}(\mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\, n_0\otimes n_0)\|^2. \end{aligned} \tag{4.40}$$

Using that $(n_0\otimes n_0)^2 = (n_0\otimes n_0)$ we deduce

$$\begin{aligned}\|\mathrm{skew}(\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0)\|^2 &= \frac{1}{4}\left\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0, \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\, n_0\otimes n_0\right\rangle - \frac{1}{4}\left\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0, n_0\otimes n_0\, \mathcal{K}_{\overline{Q}^\natural_{e,h}}\right\rangle\\ &\quad - \frac{1}{4}\left\langle n_0\otimes n_0\, \mathcal{K}_{\overline{Q}^\natural_{e,h}}, \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}\, n_0\otimes n_0\right\rangle + \frac{1}{4}\left\langle n_0\otimes n_0\, \mathcal{K}_{\overline{Q}^\natural_{e,h}}, n_0\otimes n_0\, \mathcal{K}_{\overline{Q}^\natural_{e,h}}\right\rangle\\ &= \frac{1}{2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2\,. \end{aligned} \tag{4.41}$$

We have as well

$$\begin{aligned}&2\langle \mathrm{skew}\mathcal{K}_{\overline{Q}^\natural_{e,h}}, \mathrm{skew}((0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})\rangle = 2\,\frac{(b_2-b_1)}{(b_1+b_2)}\left\langle \mathrm{skew}\mathcal{K}_{\overline{Q}^\natural_{e,h}}, \mathrm{skew}(\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0)\right\rangle\\ &\quad= \frac{(b_2-b_1)}{2(b_1+b_2)}\langle \mathcal{K}_{\overline{Q}^\natural_{e,h}}, \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0\rangle - \frac{(b_2-b_1)}{2(b_1+b_2)}\langle \mathcal{K}_{\overline{Q}^\natural_{e,h}}, n_0\otimes n_0\, \mathcal{K}_{\overline{Q}^\natural_{e,h}}\rangle\\ &\qquad - \frac{(b_2-b_1)}{2(b_1+b_2)}\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}, \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0\rangle + \frac{(b_2-b_1)}{2(b_1+b_2)}\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}}, n_0\otimes n_0\, \mathcal{K}_{\overline{Q}^\natural_{e,h}}\rangle = -\frac{(b_2-b_1)}{(b_1+b_2)}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2\,, \end{aligned} \tag{4.42}$$

and we obtain

$$\|\mathrm{skew}\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}}\|^2 = \|\mathrm{skew}\mathcal{K}_{\overline{Q}^\natural_{e,h}}\|^2 + \frac{(b_2-b_1)^2}{2(b_1+b_2)^2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2 - \frac{(b_2-b_1)}{(b_1+b_2)}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2. \tag{4.43}$$

Finally, the trace term is computed. A further needed calculation is

$$\begin{aligned}\Big[\mathrm{tr}(\mathcal{K}^{c^*}_{\overline{Q}^\natural_{e,h}})\Big]^2 &= \Big(\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}) + \mathrm{tr}\big((0|0|c)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})\big)\Big)^2\\ &= \Big(\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}) + \frac{(b_2-b_1)}{2(b_1+b_2)}\langle \mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\otimes n_0, \mathbb{1}_3\rangle - \frac{b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})\underbrace{\langle n_0\otimes n_0, \mathbb{1}_3\rangle}_{\langle n_0, n_0\rangle = 1}\Big)^2\\ &= \frac{b_1^2}{(b_1+b_3)^2}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2. \end{aligned} \tag{4.44}$$

Now we insert the above calculations in $\widetilde{W}_{\mathrm{curv}}(\mathcal{K}_{\overline{Q}^\natural_{e,h}} + (0|0|c^*)[(\mathrm{D}_x\Theta)^\natural(\eta_3)]^{-1})$ and obtain

$$\begin{aligned} W^{\mathrm{hom}}_{\mathrm{curv}} = \mu L_c^2\Big(&b_1(\|\mathrm{sym}\mathcal{K}_{\overline{Q}^\natural_{e,h}}\|^2 + \frac{(b_1-b_2)^2}{2(b_1+b_2)^2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2 + \frac{b_3^2}{(b_1+b_3)^2}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2 + \frac{b_2-b_1}{b_1+b_2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2)\\ &+ b_2(\|\mathrm{skew}\mathcal{K}_{\overline{Q}^\natural_{e,h}}\|^2 + \frac{(b_2-b_1)^2}{2(b_1-b_2)^2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2 - \frac{b_2-b_1}{b_1+b_2}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2)\\ &+ b_3\frac{b_1^2}{(b_1+b_3)^2}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2\Big)\,, \end{aligned} \tag{4.45}$$

which reduces to

$$W^{\mathrm{hom}}_{\mathrm{curv}}(\mathcal{K}_{\overline{Q}^\natural_{e,h}}) = \mu L_c^2\Big(b_1\|\mathrm{sym}\mathcal{K}_{\overline{Q}^\natural_{e,h}}\|^2 + b_2\|\mathrm{skew}\mathcal{K}_{\overline{Q}^\natural_{e,h}}\|^2 - \frac{(b_1-b_2)^2}{2(b_1+b_2)}\|\mathcal{K}^T_{\overline{Q}^\natural_{e,h}} n_0\|^2 + \frac{b_1 b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})^2\Big)\,. \tag{4.46}$$

One may apply the orthogonal decomposition of a matrix X

$$X = X^{\parallel} + X^{\perp}, \qquad X^{\parallel} := \mathrm{A}_{y_0}\, X, \qquad X^{\perp} := (\mathbb{1}_3 - \mathrm{A}_{y_0})\, X, \tag{4.47}$$

for the matrix $\mathcal{K}_{\overline{Q}^{\natural}_{e,h}}$, where $A_{y_0} = (\mathrm{D}y_0|0)[\mathrm{D}_x\Theta(0)]^{-1}$. After inserting the decomposition in the homogenised curvature energy, we get

$$\begin{aligned}
W^{\mathrm{hom}}_{\mathrm{curv}}(\mathcal{K}_{\overline{Q}^{\natural}_{e,h}}) &= \mu L_c^2\Big(b_1\|\mathrm{sym}\mathcal{K}_{\overline{Q}^{\natural}_{e,h}}\|^2 + b_2\|\mathrm{skew}\mathcal{K}_{\overline{Q}^{\natural}_{e,h}}\|^2 - \frac{(b_1-b_2)^2}{2(b_1+b_2)}\|\mathcal{K}^T_{\overline{Q}^{\natural}_{e,h}} n_0\|^2 + \frac{b_1 b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}^{\natural}_{e,h}})^2\Big)\\
&= \mu L_c^2\Big(b_1\|\mathrm{sym}\mathcal{K}^{\parallel}_{\overline{Q}^{\natural}_{e,h}}\|^2 + b_2\|\mathrm{skew}\mathcal{K}^{\parallel}_{\overline{Q}^{\natural}_{e,h}}\|^2 - \frac{(b_1-b_2)^2}{2(b_1+b_2)}\|\mathcal{K}^T_{\overline{Q}^{\natural}_{e,h}} n_0\|^2\\
&\quad + \frac{b_1 b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}^{\parallel}_{\overline{Q}^{\natural}_{e,h}})^2 + \frac{b_1+b_2}{2}\|\mathcal{K}^T_{\overline{Q}^{\natural}_{e,h}} n_0\|\Big)\\
&= \mu L_c^2\Big(b_1\|\mathrm{sym}\mathcal{K}^{\parallel}_{\overline{Q}^{\natural}_{e,h}}\|^2 + b_2\|\mathrm{skew}\mathcal{K}^{\parallel}_{\overline{Q}^{\natural}_{e,h}}\|^2 + \frac{b_1 b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}^{\parallel}_{\overline{Q}^{\natural}_{e,h}})^2 + \frac{2b_1 b_2}{b_1+b_2}\|\mathcal{K}^{\perp}_{\overline{Q}^{\natural}_{e,h}}\|\Big).
\end{aligned} \tag{4.48}$$

□

As regards the homogenised curvature energy for the following curvature energy given by the optimisation problem O4, some simplified calculations as for the optimisation problem O3 lead us to

Corollary 4.3. *The solution of the optimisation problem O3 given in* (4.10) *is given by*

$$c^* = \frac{(b_2-b_1)}{b_1+b_2}\mathcal{K}^T_{\overline{Q}_{e,0}} n_0 - \frac{2b_3}{2(b_1+b_3)}\mathrm{tr}(\mathcal{K}_{\overline{Q}_{e,0}}) n_0 \tag{4.49}$$

and the corresponding homogenised curvature energy is

$$W^{hom}_{curv}(\mathcal{K}_{\overline{Q}_{e,0}}) = \mu L_c^2\Big(b_1\|\mathrm{sym}\mathcal{K}^{\parallel}_{\overline{Q}_{e,0}}\|^2 + b_2\|\mathrm{skew}\mathcal{K}^{\parallel}_{\overline{Q}_{e,0}}\|^2 + \frac{b_1 b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}^{\parallel}_{\overline{Q}_{e,0}})^2 + \frac{2b_1 b_2}{b_1+b_2}\|\mathcal{K}^{\perp}_{\overline{Q}_{e,0}}\|\Big). \tag{4.50}$$

where $\mathcal{K}^{\parallel}_{\overline{Q}_{e,0}}$ and $\mathcal{K}^{\perp}_{\overline{Q}_{e,0}}$ represent the orthogonal decomposition of the fully reduced elastic shell bending curvature tensor $\mathcal{K}_{\overline{Q}_{e,0}}$ in the tangential plane and in the normal direction, respectively.

4.2. Γ-convergence result for the curved shell model

Together with the calculations provided in [40], we obtain for the first time in the literature the explicit form of the Cosserat shell model via Γ-convergence method given by the following theorem.

Theorem 4.4. *Assume that the initial configuration of the curved shell is defined by a continuous injective mapping $y_0 : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ which admits an extension to $\overline{\omega}$ into $C^2(\overline{\omega};\mathbb{R}^3)$ such that for*

$$\Theta(x_1,x_2,x_3) = y_0(x_1,x_2) + x_3\, n_0(x_1,x_2)$$

we have $\det[\mathrm{D}_x\Theta(0)] \geq a_0 > 0$ on $\overline{\omega}$, where a_0 is a constant, and assume that the boundary data satisfy the conditions

$$\varphi^{\natural}_d = \varphi_d\big|_{\Gamma_1} \text{(in the sense of traces) for } \ \varphi_d \in \mathrm{H}^1(\Omega_1;\mathbb{R}^3). \tag{4.51}$$

Let the constitutive parameters satisfy

$$\mu > 0, \qquad \kappa > 0, \qquad \mu_c > 0, \qquad a_1 > 0, \qquad a_2 > 0, \qquad a_3 > 0\,. \tag{4.52}$$

Then, for any sequence $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \in X$ *such that* $(\varphi^\natural_{h_j}, \overline{Q}^\natural_{e,h_j}) \to (\varphi_0, \overline{Q}_{e,0})$ *as* $h_j \to 0$*, the sequence of functionals* $\mathcal{J}_{h_j}\colon X \to \overline{\mathbb{R}}$ *from* (4.7) Γ*-converges to the limit energy functional* $\mathcal{J}_0\colon X \to \overline{\mathbb{R}}$ *defined by*

$$\mathcal{J}_0(m,\overline{Q}_{e,0}) = \begin{cases} \displaystyle\int_\omega [W^{\rm hom}_{\rm mp}(\mathcal{E}_{m,\overline{Q}_{e,0}}) + \widetilde{W}^{\rm hom}_{\rm curv}(\mathcal{K}_{\overline{Q}_{e,0}})] \det(\mathrm{D}y_0|n_0)\, d\omega & \textit{if} \quad (m,\overline{Q}_{e,0}) \in \mathcal{S}'_\omega, \\ +\infty & \textit{else in } X, \end{cases} \tag{4.53}$$

where

$$\begin{aligned} m(x_1,x_2) &:= \varphi_0(x_1,x_2) = \lim_{h_j\to 0} \varphi^\natural_{h_j}(x_1,x_2,\frac{1}{h_j}x_3), \qquad \overline{Q}_{e,0}(x_1,x_2) = \lim_{h_j\to 0} \overline{Q}^\natural_{e,h_j}(x_1,x_2,\frac{1}{h_j}x_3), \\ \mathcal{E}_{m,\overline{Q}_{e,0}} &= (\overline{Q}^T_{e,0}\mathrm{D}m - \mathrm{D}y_0|0)[\mathrm{D}_x\Theta(0)]^{-1}, \\ \mathcal{K}_{\overline{Q}_{e,0}} &= \Big(\mathrm{axl}(\overline{Q}^T_{e,0}\,\partial_{x_1}\overline{Q}_{e,0})\,|\,\mathrm{axl}(\overline{Q}^T_{e,0}\,\partial_{x_2}\overline{Q}_{e,0})\,|0\Big)[\mathrm{D}_x\Theta(0)\,]^{-1} \not\in \mathrm{Sym}(3)\,, \end{aligned} \tag{4.54}$$

and

$$\begin{aligned} W^{\rm hom}_{\rm mp}(\mathcal{E}_{m,\overline{Q}_{e,0}}) =&\ \mu\,\|\mathrm{sym}\ \mathcal{E}^\parallel_{m,\overline{Q}_{e,0}}\|^2 + \mu_c\,\|\mathrm{skew}\ \mathcal{E}^\parallel_{m,\overline{Q}_{e,0}}\|^2 + \frac{\lambda\,\mu}{\lambda+2\,\mu}\,\big[\mathrm{tr}(\mathcal{E}^\parallel_{m,\overline{Q}_{e,0}})\big]^2 + \frac{2\,\mu\,\mu_c}{\mu_c+\mu}\|\mathcal{E}^T_{m,\overline{Q}_{e,0}}n_0\|^2 \\ =&\ W_{\rm shell}(\mathcal{E}^\parallel_{m,\overline{Q}_{e,0}}) + \frac{2\,\mu\,\mu_c}{\mu_c+\mu}\|\mathcal{E}^\perp_{m,\overline{Q}_{e,0}}\|^2, \\ \widetilde{W}^{\rm hom}_{\rm curv}(\mathcal{K}_{\overline{Q}_{e,0}}) =&\ \inf_{A\in\mathfrak{so}(3)} \widetilde{W}_{\rm curv}\Big(\mathrm{axl}(\overline{Q}^T_{e,0}\,\partial_{\eta_1}\overline{Q}_{e,0})\,|\,\mathrm{axl}(\overline{Q}^T_{e,0}\,\partial_{\eta_2}\overline{Q}_{e,0})\,|\,\mathrm{axl}(A)\Big)[(\mathrm{D}_x\Theta)^\natural(0)]^{-1} \\ =&\ \mu L_c^2\Big(b_1\|\mathrm{sym}\mathcal{K}^\parallel_{\overline{Q}_{e,0}}\|^2 + b_2\|\mathrm{skew}\mathcal{K}^\parallel_{\overline{Q}_{e,0}}\|^2 + \frac{b_1b_3}{(b_1+b_3)}\mathrm{tr}(\mathcal{K}^\parallel_{\overline{Q}_{e,0}})^2 + \frac{2\,b_1b_2}{b_1+b_2}\|\mathcal{K}^\perp_{\overline{Q}_{e,0}}\|\Big)\,. \end{aligned} \tag{4.55}$$

Proof. The proof is completely similar to the proof provided in [40], where only some implicit properties of the homogenised curvature energy were used and not its explicit form. □

5. Conclusion

The present paper gives the explicit calculation of the homogenised curvature energy. This explicit form was not directly necessary in order to prove the previous Γ-convergence result, some qualitative properties of $\widetilde{W}^{\rm hom,\natural}_{\rm curv}(\mathcal{K}_{\overline{Q}^\natural_{e,h}})$ and $\widetilde{W}^{\rm hom}_{\rm curv}(\mathcal{K}_{\overline{Q}_{e,0}})$ being enough in the proof of the Γ-convergence result. However, the final Γ-convergence model has to be written in a explicit form, all the explicit calculations being provided in this paper.

A comparison between (3.23) and (5.2) shows that the homogenised flat curvature energy can thus be obtained from the curved one, and that Theorem 3.2 may be seen as a corollary of Theorem 4.4. Indeed, let us assume that in the homogenised energy which we obtained in (4.48) we have $\mathrm{D}\Theta = \mathbb{1}_3, \mathrm{D}y_0 = \mathbb{1}$ and $n_0 = [\mathrm{D}_x\Theta(0)]e_3 = e_3$, which corresponds to the flat shell case. Then $\overline{Q}_{e,0} = \overline{R}_0$, $\mathcal{K}_{\overline{Q}_{e,0}} = \mathcal{K}^{\rm plate}_{\overline{R}_0}$ and

$$\mathcal{K}_{\overline{Q}_{e,0}} = \left(\begin{pmatrix} \Gamma_\square & & 0 \\ & & 0 \\ \hline \Gamma_{31} & \Gamma_{32} & 0 \end{pmatrix}\right)[(\mathrm{D}_x\Theta)^\natural]^{-1} = \mathcal{K}^{\rm plate}_{\overline{R}_0}, \quad \text{with} \quad \Gamma_\square = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \tag{5.1}$$

and we have

$$W^{\rm hom}_{\rm curv}(\Gamma) = \mu L_c^2\Big(b_1\|\mathrm{sym}\Gamma_\square\|^2 + b_2\|\mathrm{skew}\Gamma_\square\|^2 + \frac{b_1b_3}{(b_1+b_3)}\mathrm{tr}(\Gamma_\square)^2 + \frac{2b_1b_2}{b_1+b_2}\left\|\begin{pmatrix}\Gamma_{31}\\ \Gamma_{32}\end{pmatrix}\right\|^2\Big)\,, \tag{5.2}$$

and we rediscover the homogenised curvature energy for an initial flat configuration from Theorem 3.2.

In conclusion, the present paper completes the calculations of the membrane-like model constructed via Γ-convergence for flat and curved initial configuration of the shell given for the first time in literature the explicit form of the Γ-limit for both situations.

In [5], by using a method which extends the reduction procedure from classical elasticity to the case of Cosserat shells, Bîrsan has obtained a Cosserat shell by considering a general ansatz. For the particular case of a quadratic ansatz for the deformation map and skipping higher-order terms, the membrane term of order $O(h)$ from the Birsan's model [5] coincides with the homogenised membrane energy determined by us in [40], i.e. in both models the harmonic mean $\dfrac{2\mu\,\mu_c}{\mu+\mu_c}$ of μ and μ_c is present. We note that in the model constructed in [20] the algebraic mean of μ and μ_c substitutes the role of the harmonic mean from the model given in [5] and by the Γ-convergence model in [40].

However, a comparison between the curvature energy obtained in the current paper as part of the Γ-limit and the curvature energy obtained using other methods [5,20] shows that the weight of the energy term $\|\mathcal{K}^{\perp}_{e,s}\|^2$ is different as follows

- Derivation approach [20] as well as in the model given in [5]: the algebraic mean of b_1 and b_2, i.e. $\dfrac{b_1+b_2}{2}$;
- Γ-convergence: the harmonic mean of b_1 and b_2, i.e. $\dfrac{2\,b_1 b_2}{b_1+b_2}$.

From mathematical point of view, different weights do not produce differences. These differences were discussed in the numerical analysis given in [30,41].

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest We declare that we have no conflict of interest.

Consent for publication Our manuscript has not been published before and is not under consideration for publication elsewhere. Herein, we give our consent for the publication.

Ethics approval and consent to participate All studies are based on previous published researches and do not involve ethics. The study does not include human or animal subjects. We claim that the manuscript has been read and approved by all named authors.

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