THE LATTICE STRUCTURE ASSOCIATED TO COMPLETE HYPERGROUPS

STRUCTURA LATICEALĂ ASOCIATĂ HIPERGRUPURILOR COMPLETE

SONEA Cristina Andromeda[1]*

[*]Corresponding author e-mail: soneaandromeda@yahoo.com

Abstract. *According with representation theorem for complete hypergroups, we can determine a relationship between the lattice of subgroups associated with a group and the lattice of subhypergroups associated with complete hypergroups. Also, we present some examples to illustrate this connection.*
Keywords: lattice, complete hypergroup, closed subhypergroup

Rezumat. *Conform teoremei de caracterizare a hipergrupurilor complete, putem determina o relație dintre laticea subgrupurilor asociate unui grup si laticea subhipergrupurilor associate hipergrupului complet corespunzător. De asemenea, prezentăm exemple ce ilustrează această conexiune.*
Cuvinte cheie: *latice, hipergrup complet, subhipergrup închis*

INTRODUCTION

The hypergroup theory was born in 1934, when F. Marty defined the notion of a hyperoperation in the following way. If $H \neq \varnothing$ is a nonempty set and H is endowed with a hyperoperation $"\circ": H \times H \to P^*(H)$ which represents a generalization of composition law from group theory, because if we compose two elements in H the result is a set, not always a singleton as in group theory. Starting with hypergroupoid $(H, \circ)$ was developed many other definitions which make a connection with results from group theory. G. Birkhoff published the first edition of the book Lattice Theory, when he analyzed the properties of algebraic structures using the lattice of sub algebraic structures. In hypergroup theory, it is a difficult problem to determine the lattice structure of subhypergroups. Bayrack investigate the properties of closed, invertible, ultraclosed and conjugable subhypergroups classes. He considered that hyperproduct of subhypergroups becomes an operation on the set of subhypergroups. We denoted by $Sub(H), CSub(H), ISub(H), ConSub(H)$ the sets of all subhypergroups, closed subhypergroups, invertible subhypergroups, ultraclosed subhypergroups and conjugable subhypergroups of H In this article, we studied an important class of hypergroups, i.e. complete hypergroups. Sonea (2020) and Cristea (2002)

[1]Iaşi University of Life Sciences, Iași, Romania

calculate the commutativity degree and the fuzzy grade of them. In this paper, we analyze the subhypergroups of complete hypergroups and we prove that all the subhypergroups formed a lattice structure associated to complete hypergroup but using some particularly conditions.

PRELIMINARIES

In this section, we present some basic notions from hypergroup theory and lattice theory.

Definition 1
Let $H \neq \emptyset$ be a nonempty set and $"\circ": H \times H \rightarrow P^*(H)$ is a hyperoperation, where $P^*(H)$ is the set of all nonempty subsets of $H.$ Then $(H,\circ)$ is a hypergoupoid.

Remark 1
If A and B are nonempty sets of $H,$ then $A \circ B = \bigcup_{\substack{a\in A \\ b\in B}} a \circ b.$

Definition 2
i) A semihypergroup is a hypergroupoid $(H,\circ)$ such that for any $(a,b,c) \in H^3$, $(a \circ b) \circ c = a \circ (b \circ c).$
ii) A quasihypergroup is a hypergroupoid $(H,\circ)$ which satisfies the reproductive law: for any $a \in H: H \circ a = a \circ H = H.$
iii) A hypergroup is a semihypergroup which is also a quasihypergroup.

Definition 3
Let $(H,\circ)$ be a hypergroupoid. An element $e \in H$ is called an identity or unit if, for any $a \in H, a \in a \circ e \bigcap e \circ a.$

Definition 4
Let $(H,\circ)$ be a hypergroupoid, endowed with at least one identity. An element $a_0 \in H$ is called an inverse of $a \in H$ if there is an identity $e \in H,$ such that $e \in a_0 \circ a \bigcap a \circ a_0.$

Definition 5
A hypergroup $(H,\circ)$ is called a regular hypergroup, if it has at least one identity element and all elements of H have at least one inverse.

Definition 6

A regular hypergroup $(H,\circ)$ is called reversible if for all $x,y,z\in H$ such that $x\in y\circ z$ it follows that $y\in x\circ z'$ and $z\in y'\circ x$ for some $y'\in i(y),\ z'\in i(z),$ where $i(x)$ represents the sets of all inverse of element x.

Theorem 1 (Characterization theorem of complete hypergroups)

A hypergroup $(H,\circ)$ is complete, if and only if there exists nonempty subsets A_g of $H,$ for all $g\in G$ such that $H=\bigcup_{g\in G} A_g$. G and A_g satisfy the following conditions:

1. $(G,.)$ is a group.
2. For any $g_1\neq g_2, g_1, g_2\in G: A_{g_1}\bigcap A_{g_2}=\varnothing.$
3. If $(a,b)\in A_{g_1}\times A_{g_2}$, then $a\circ b=A_{g_1g_2}$.

Definition 7

A lattice represents a partially ordered set in which every pair of elements has a unique supremum (also called a least upperbound or join)and a unique infimum (also called a greatest lowerbound or meet).

RESULTS AND DISCUSSIONS

Proposition 1

Let $(H,\circ)$ be a complete hypergroup and let G be a finite group, that appears in the representation theorem of the complete hypergroup H. If P is a subgroup of G, then

$$K=\bigcup_{k\in P} A_k$$

Is a subhypergroup of H.

Proof:

We consider $(H,\circ)$ be a complete hypergroup and P a subgroup of (G,.), where G characterized the complete hypergroup and $K=\bigcup_{k\in P} A_k$ is a subhypergroup of $H,$ if and only if the next relations holds on: 1) $a\circ b\subseteq K,$ for any $a,b\in K;$ 2) $a\circ K=K\circ a=K,$ for any $a\in K.$

The first condition becomes to for each a and b from K there exists and unique $k_1,k_2\in K$ such that $a\circ b=A_{k_1k_2}$. But K is a subgroup of G, which implies that

$k_1k_2 \in P$ and we get $A_{k_1k_2} \subseteq K$. The second condition could be written as it follows

$$a \circ K = \bigcup_{b\in K}(a \circ b) = \bigcup_{i=1}^{n} A_{kk_i} = \bigcup_{i=1}^{n} A_{k_i},$$

$$K \circ a = \bigcup_{b\in K}(b \circ a) = \bigcup_{i=1}^{n} A_{k_ik} = \bigcup_{i=1}^{n} A_{k_i},$$

Were $a \in A_k, k \in P, |P| = n, n \in \mathbb{N}^*$. To prove the last equality, we consider the function $f : P \to P, f(k_i) = kk_i, i \in \{1, 2, ..., n\}, k \in P$. We remark that f is an one to one function: $f(k_i) = f(k_j)$ implies $kk_i = kk_j$. We multiply by left with inverse of k and we obtain $k_i = k_j$. Moreover, the cardinality of P is finite, so f is an onto function. Means that for any $k_j \in P$, there is $k_i \in P$ such that $f(k_i) = k_j$. So, $\bigcup_{i=1}^{n} A_{kk_i} = \bigcup_{i=1}^{n} A_{k_i}$. In the same manner, we show that $K \circ a = \bigcup_{i=1}^{n} A_{k_i}$. In conclusion, if P is a subgroup of group $(G,.)$, then K is a subgroup of complete hypergroup $(H,\circ)$.

Proposition 2

Let $(H,\circ)$ be a complete hypergroup and let G be a finite group, that appears in the representation theorem of the complete hypergroup H. If P is a nonempty set of G, such that $K = \bigcup_{k\in P} A_k$ is a subhypergroup of H, then P is a subgroup of G.

Proof:

We consider $P \neq \emptyset$, $P \subset G$ such that $K = \bigcup_{k\in P} A_k$ is a subhypergroup of H.

As K is subhypergroup of complete hypergroup H, results that for any $a, b \in K$ we have $a \circ b \subseteq K$ and for any $a \in K$ we get $a \circ K = K \circ a = K$. The first condition could be write thus: for any $a, b \in K$ there is $k_i, k_j \in P; i, j \in \{1, 2, ..., n\}$, where $n = |P|, n \in \mathbb{N}^*$ so as to $A_{k_ik_j} \subseteq K$ results that $k_ik_j \in P$.

The relation hold on for any a and b, whichimplies that for any $k_i, k_j \in P$ results that $k_ik_j \in P$. In the following, we prove that for any $k \in P$, the inverse $k^{-1} \in P.$ First, we have to show that the heart of H belongs to K. Therefore

$$a \in K = K \circ a = a \circ K$$

results that exists $b \in K$ such that $a \in a \circ b.$ Using the characterization theorem we have $A_k \subseteq A_{kk_1}$, where $a \in A_k, b \in A_{k_1}; k, k_1 \in P$ which implies that $k = kk_1$ so $k_1 = e,$ where e is the neutral element of group G. Therefore $b \in A_e = \omega_H$. In conclusion $\omega_H \subseteq K.$ Let $a_0 \in \omega_H \subseteq K$, so exists $a' \in K$ such that

$$a_0 \in a \circ a' \Rightarrow A_e = A_{kt} \Rightarrow e = kt \Rightarrow t = k^{-1} \in P,$$

where $a' \in A_t \subseteq K,$ results that $t \in P.$ So, for any $k \in P$ there is $k^{-1} \in P.$ In conclusion, P is subgroup of $G.$

Theorem2

Let $(H, \circ)$ be a complete hypergroup and let G be a finite group, that appears in the representation theorem of the complete hypergroup $H.$ Then P is a subgroup of group G if and only of $K = \bigcup_{k \in P} A_k$ is a subhypergroup of complete hypergroup $H.$

Proof:
The proof is straight forward.
In the subhypergroup theory, we have a kind of subhypergroups as: closed, ultra-closed, conjugable, invertible and so on. In the following, we want to prove that the subhypergroups determined in the Theorem 1 are closed.

Definition 8

Let $(H, \circ)$ be a hypergroup and K subhypergroup of $H.$ We say that K is a closed hypergroup if the following conditions holds on:

$$a \in b \circ x \Rightarrow x \in K;$$

$$a \in x \circ b \Rightarrow x \in K.$$

Proposition 3

Let $(H, \circ)$ be a complete hypergroup and let G be a finite group, that appears in the representation theorem of the complete hypergroup $H.$ Let $K = \bigcup_{k \in P} A_k$ be a subhypergroup of $H,$ where P is a subgroup of group $G.$ Then K is a closed subhypergroup.

Proof:

We consider $a,b \in K$ and $x \in H.$ According with characterization theorem, results that exists $k_1, k_2 \in P, g \in G.$ such that $a \in A_{k_1}$, $b \in A_{k_2}$, $x \in A_g$. Therefore, the conditions gives by definition 8 could be write in the following way

$$A_{k_1} \subseteq A_{k_2 g} \Rightarrow k_1 = k_2 g \Rightarrow g = k_2^{-1} k_1 \in P$$
$$A_{k_1} \subseteq A_{g k_2} \Rightarrow k_1 = g k_2 \Rightarrow g = k_1 k_2^{-1} \in P,$$

because P is subgroup of group $G.$ So $x \in K.$

In (Bayrack, 2019), it is proved that the set of invertible subhypergroups of a hypergroup determine a distributive lattice and the set of ultra-closed subhypergroups formed a modular lattice.

Proposition 4

Let $(H, \circ)$ be a complete hypergroup and subhypergroups $K_i, K_j \in CSub(H),$ where $i, j \in \{1, 2, ..., |L(G)|\}$. Then $K_i \circ K_j = K_j \circ K_i$ if and only if $K_i \circ K_j, K_j \circ K_i \in CSub(H).$

Proof:

We suppose that $K_i \circ K_j = K_j \circ K_i$, where $i, j \in \{1, 2, ..., |L(G)|\}$. Let $a \in K_i \circ K_j$ results that exists $k_i \in K_i, k_j \in K_j$ such that $a \in k_i \circ k_j$. As H is a complete hypergroup, implies that H is a reversible hypergroup. So, there is $k'_i \in i(k_i)$ thus $k_j \in k'_i \circ a.$ According to **theorem 1**, follows that there is P_i subgroup of G such that $K_i = \bigcup_{p_i \in P_i} A_{p_i}$. But $k_i \in K_i,$ which implies that exists $p_i \in P_i$ with property $k_i \in A_{p_i}$. Using the **Characterization theorem for complete hypergroups**, we can asserts that $k_i \circ k^{-1}{}_i = A_e,$ where e is the neutral element of group G It leads to $A_{p_i g_i} = A_e, k_i^{-1} \in A_{g_i}, g_i \in G.$ We get $p_i g_i = e \Rightarrow g_i = p_i^{-1},$ but P_i is subgroup of the group $G,$ so $g_i \in P_i$ and $k_i^{-1} \in K_i.$ Therefore,

$k_j \in k_i' \circ a \subseteq K_i \circ a.$ Then, $K_i \circ K_j = K_i \circ K_j \circ k_j \subseteq K_i \circ K_j \circ K_i \circ a =$
$= K_i \circ K_j \circ a.$

We have $K_i \circ K_j \circ a \subseteq K_i \circ K_j \circ k_i \circ k_j = K_j \circ K_i \circ k_i \circ k_j = = K_j \circ K_i \circ k_j = K_i \circ K_j$. Therefore $(K_i \circ K_j) \circ a = K_i \circ K_j$. We notice that for any $a, b \in K_i \circ K_j$

$$a \circ b \subseteq K_i \circ K_j \circ K_i \circ K_j = K_i \circ K_j.$$

So we assert that $K_i \circ K_j \in Sub(H)$. $K_i \circ K_j$ is closed subhypergroup if and only if for any $a, b \in K_i \circ K_j, x \in H, a \in b \circ x$ and $a \in x \circ b$, results $x \in K_i \circ K_j$. We analyze the first case, and for the second the proof is similar. If $a \in b \circ x$ and H is a reversible hypergroup, results that there is $b' \in i(b)$ such that $x \in b' \circ a$. As $b \in K_i \circ K_j$ follows that $b' \in K_i \circ K_j$. We showed that $K_i \circ K_j \in Sub(H)$ which leads to $x \in K_i \circ K_j$. In conclusion, $K_i \circ K_j \in CSub(H)$.

Now, if $K_i \circ K_j, K_j \circ K_i \in CSub(H)$, then $K_i \circ K_j = K_j \circ K_i$. We show that $K_j \circ K_i \circ K_j \subseteq K_i \circ K_j$. Let $a \in K_j \circ K_i \circ K_j, K_i \circ K_j \in CSub(H)$, results that $(K_i \circ K_j) \circ (K_i \circ K_j) = K_i \circ K_j$. So, $K_i \circ a \subseteq K_i \circ K_j$. Because $K_i \circ a \neq \emptyset$, means that exists $x \in K_i \circ K_j$ such that $x \in K_i \circ a = \bigcup_{k \in K_i} (k \circ a)$. Hence exists $k_0 \in K_i$ thus $x \in k_0 \circ a$. But H is a reversible hypergroup, so there is $k_0' \in i(k)$ with

$$a \in k_0' \circ x \subseteq K_i \circ x \subseteq K_i \circ K_i \circ K_j = K_i \circ K_j.$$

Consequently $K_j \circ K_i \circ K_j \subseteq K_i \circ K_j$. In the following, we prove that $K_j \circ K_i \subseteq K_i \circ K_j$. Let $b \in K_j \circ K_i$, implies that $b \circ K_j \subseteq K_j \circ K_i \circ K_j \subseteq K_i \circ K_j$. We know that $b \circ K_j \neq \emptyset$ and proceeding in a similar way like in the case $K_i \circ a \neq \emptyset$, we have $y \in K_i \circ K_j$. So $b \in x \circ K_j \subseteq K_i \circ K_j \circ K_j = K_i \circ K_j$, then $K_j \circ K_i \subseteq K_i \circ K_j$. Similarly we obtain the relation $K_i \circ K_j \subseteq K_j \circ K_i$. In conclusion, $K_i \circ K_j = K_j \circ K_i$ if and only if $K_i \circ K_j, K_j \circ K_i \in CSub(H)$.

Remark2

Let H be a complete hypergroup and K subhypergroup of H, then $\omega_H \subseteq K$.

Theorem 3

Let $(H,\circ)$ be a complete hypergroup and let $\{K_i\}_{i\geq 1} \in CSub(H)$ subhypergroups which satisfies the proposition 3. Then $(CSub(H),\vee,\wedge)$ is a lattice, where $K_i \vee K_j = K_i \circ K_j$ and $K_i \wedge K_j = K_i \bigcap K_j$, for any $K_i, K_j \in CSub(H)$.

Proof:
According to theorem 1 and remark 2, we notice that for any $K_i, K_j \in CSub(H)$. implies that $K_i \bigcap K_j \in CSub(H)$. Also, the condition $K_i \vee K_j \in CSub(H)$ holds on. In the following, we prove that $K_i \bigcup K_j \subseteq K_i \circ K_j$. Using theorem 2, there is P_i, P_j subgroups of the group G, such that $K_i = \bigcup\limits_{p\in P_i} A_p$, $K_j = \bigcup\limits_{g\in P_j} A_g$.

$$K_i \circ K_j = \bigcup_{\substack{a\in K_i \\ b\in K_j}} (a\circ b) = \bigcup_{\substack{p\in P_i \\ g\in P_j}} A_{pg}.$$

We consider $h = e$ and we get $K_j \subseteq K_i \circ K_j$. For $g = e$, we have $K_i \subseteq K_i \circ K_j$, where e is the neutral element of the group G. Therefore, $K_i \bigcup K_j \subseteq K_i \circ K_j$. In the following, we prove that the $K_i \circ K_j$ is the smallest closed subhypergroup containing on K_i respectively K_j.
Let $K_s \in CSub(H)$ such that $K_i \subseteq K_s, K_j \subseteq K_s$, then $K_i \circ K_j \subseteq K_s \circ K_s = K_s$.
In conclusion, $K_i \vee K_j = K_i \circ K_j, K_i \wedge K_j = K_i \bigcap K_j$.

Theorem 4

Let $(H,\circ)$ be a complete hypergroup and $(CSub(H),\vee,\wedge)$ the lattice associated to H .Then $(CSub(H),\vee,\wedge)$ is a modular lattice.

Proof:
Let $K_i, K_j, K_s \in CSub(H)$ such that $K_i \supseteq K_j$. Any lattice satisfies the modular inequality:

$$K_j \vee (K_i \wedge K_s) \subseteq K_i \wedge (K_j \vee K_s).$$

Let $x \in K_i \wedge (K_j \vee K_s)$ results that $x \in K_i$ and $x \in K_j \circ K_s$, means that exists $b \in K_j$ and $c \in K_s$, thus $x \in b\circ c$. As H is a reversible hypergroup, implies that exists $b' \in i(b)$ such that for any $c \in b'\circ x$. In the proof of proposition 9, we get the following: if $b \in K_j, K_j \in Sub(H)$, then $b' \in K_j$. So, $c \in K_j \circ K_i$ which implies $c \in K_i$. Therefore, $x \in b \circ c \subseteq K_j \circ (K_i \cap K_s)$.

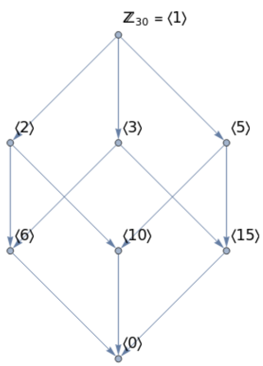
We get $K_i \wedge (K_j \vee K_s) \subseteq K_j \vee (K_i \wedge K_s)$.

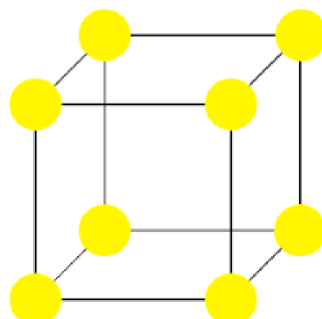
In conclusion, we have $K_j \vee (K_i \wedge K_s) = K_i \wedge (K_j \vee K_s)$.

EXAMPLES

In this section, we present some connections between the lattice of subhypergroups associated to complete hypergroup and lattice structure of chemical elements.

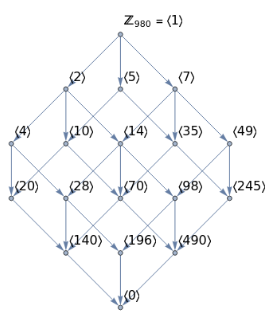
We consider the complete hypergroup H with characterization group $G = (Z_{30}, +)$, which is a cyclic group. The elements of G has the following form: $Z_{30} = \{\hat{0}, \hat{1}, ..., 29\}$, where $k -$ represents all numbers that have the rest k when divides 30. Also, in molecular symmetry it is very known the lattice crystal. So, we obtain.



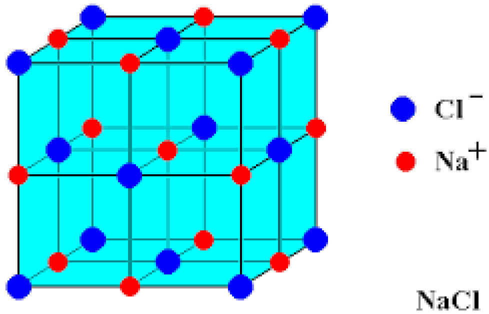


SubgroupsLattice of Z_{30} *Crystal Lattice*

We consider the complete hypergroup H with characterization group $G = (Z_{980}, +)$, which is a cyclic group and the crystal structure of sodium chlorhide.



SubgroupsLattice of Z_{980}



Crystal structure of sodium chlorhide

REFERENCES

1. **Bayrack D., 2019 -** *The lattice of subhypergroups of a hypergroups*. Eskisehir Teknik Universitesi Bilim ve Teknoloji Dergisi B - Teorik Bilimler.
2. **Birkhoff G., 1940 -** *Lattice Theory*. First edition, Amer. Math. Soc., Providence, R.I. v+155.
3. **Birkhoff G., 1961 -** *Lattice Theory*. Revised ed. Amer. Math. Soc. Colloquim Publications,vol. 25, Amer. Math. Soc., Providence, R.I. xiii+283.
4. **Birkhoff G., 1979 -** *Lattice Theory*. Corrected reprint of the 1967 third edition. Amer. Math.Soc. Colloquim Publications, vol. 25, Amer. Math. Soc., Providence, R.I. vi+418.
5. **Corsini P., Leoreanu V., 2003 -** *Applications of hyperstructures theory*. Kluwer Academic Publishers Boston/Dordrechtt/ London.
6. **Cristea I., 2002 -** *Complete hypergroups, 1 - hypergroups and fuzzy sets*. An. St. Univ. Ovidius Constanta, vol. 10(2), 25, 38.
7. **Graetzer G., 2011 -** *Lattice Theory*. Foundation, Birkhauser.
8. **Heidari D., Davvaz B., 2011 -** *On ordered hyperstructures*. U.P.B. Sci. Bull. Series A, vol. 73, Iss. 2.
9. **Nakano T., 1967 -** *Rings and partly ordered systems.* Math. Zeitschrift, 99.
10. **Sonea A., Cristea I., 2020 -** *The class equation and the commutativity degree for complete hypergroups*. Mathematics, 8(12), 2253.