

Article

Fuzzy Aspects Associated with Biological Inheritance

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Abstract

Genetics explores the mechanisms of heredity and trait variation across organisms, with foundational principles established by Gregor Mendel through experiments on monohybrid and dihybrid crosses. The mathematical framework of hypergroups can effectively describe these classical genetic models. This study examines the interaction between genetic hybridization models and fuzzy set theory. It focuses on the fuzzy function that relates to phenotype classes made from simple dominance in dihybrid, trihybrid, and polyhybrid crosses. The methods use fuzzy logic to model phenotype distributions. The results show a clear link between the structure of the fuzzy function and the number of distinct phenotype classes in each hybridization case. This article presents a general form for the fuzzy function, and it always follows the same order relation. The number of phenotypes in each class determines this relation. Therefore, each class is associated with a string that serves as a row in the matrix describing the respective hybridization. Studies have shown that the eigenvalues of this matrix coincide with its elements.

Keywords: fuzzy function; genetics; biological inheritance; hibridization; hypergroup**MSC:** 20N20

1. Introduction

The theory of hypergroups originated in 1934 during a Scandinavian Congress, where F. Marty introduced the concept of the hypergroup [1]. A hypergroup is an algebraic structure similar to a group, but the composition of two elements results in a nonempty set rather than a single element. This theory has evolved significantly both theoretically and practically, with numerous applications across various fields, such as biology [2,3], genetics [4–6], graph theory [7,8], chemistry [9], dependence relations theory [10], automata theory [11,12], cryptography [13], and fuzzy sets [14–17]. Direct applications of hyperstructures in genetics appear in [2], where the authors analyzed hypergroups formed by genotypes in various situations, such as a hypothetical cross involving n different traits, simple dominance, dominant gene epistasis in dog coat color, supplementary gene for anthocyanin pigmentation in flowers, inhibitory gene in rice leaves, complementary gene in some rice varieties, and both supplementary and complementary genes in seed coat color. Sonea et al. further developed this topic in their 2023 study [6], where they determined the distribution of phenotypes in polyhybrid crosses under simple dominance. The present paper continues this line of research by focusing on the analysis of the fuzzy function in the context of simple dominance. To ensure the genetic concepts used throughout the paper are accessible, we will begin by presenting several theoretical aspects from the field of genetics. Genetics is the branch of biology that studies the heredity and variability of living organisms, explaining the mechanisms responsible for recording, modifying, and transmitting



Academic Editor: Artem A. Lopatin

Received: 30 October 2025

Revised: 23 November 2025

Accepted: 26 November 2025

Published: 1 December 2025

Citation: Sonea, A.P. Fuzzy Aspects Associated with Biological Inheritance. *Mathematics* **2025**, *13*, 3847. <https://doi.org/10.3390/math13233847>

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hereditary information from one generation to another, as well as the complex interactions between genotype and environment. The founder of modern genetics is considered to be the biologist and mathematician Gregor Mendel, who conducted hybridization experiments on several plant species, including pea, maize, and bean. Based on these studies, Mendel formulated the theory of hereditary factors, according to which each trait is determined by a specific material unit called a gene, located in the cell nucleus and transmitted to the offspring through gametes. By analyzing the manifestation of traits in hybrid generations $F_1, \dots, F_n$, Mendel derived general laws of heredity, which later became known as Mendel's Laws. Hybridization refers to the process by which hybrid organisms are created through the crossing of two or more parents that differ in one or more traits. The hybrid combines the characters inherited from both parental forms, denoted conventionally as P_1 and P_2 , while the hybrid generations are represented as $F_1, \dots, F_n$. When the parents differ by a single pair of traits, the process is known as monohybridization; when they differ by two or more pairs, it is called dihybridization or polyhybridization. Hybridizations represent an important topic in the field of horticulture. Numerous studies focus on different varieties of fruit trees, such as pear, apple, and cherry [18,19]. Organisms possessing only one type of allele are termed homozygous (AA is dominant, aa is recessive), while those carrying both allelic variants are heterozygous (Aa). In heterozygotes, the dominant character (A) is phenotypically expressed, whereas the recessive character (a) remains hidden. Consequently, Mendel introduced two fundamental concepts: genotype, the totality of hereditary factors (genes) contained in an organism, and phenotype, the ensemble of morphological, physiological, biochemical, and behavioral traits resulting from the expression of the genotype [20]. This paper represents a continuation of the study initiated in [6], expanding the analysis of the fuzzy function associated with phenotype classes in the cases of dihybrid, trihybrid, and polyhybrid crosses. Genetic processes such as dihybrid and trihybrid recombinations naturally involve uncertainty arising from incomplete dominance, epistatic effects, and partial information on allele interactions. Classical algebraic genetic models describe the structure of recombination but do not incorporate uncertainty. Our motivation is to develop a mathematical framework that quantifies this uncertainty by introducing a fuzzy function associated with the hyperoperation of the genetic model, providing a continuous description of genetic contributions and enabling the study of stabilization phenomena in recombination systems. Also, the number of resulting phenotypes is a key component in the computation of the fuzzy function. The structure of the paper is as follows: Section 1 presents the background of the topic and the motivation of the study. Section 2 introduces the basic notions of hypergroup theory and describes the algorithm for computing fuzzy functions and determining the fuzzy degree of a hypergroupoid. Section 3 contains the computation of the fuzzy degree in the cases of dihybridization and trihybridization processes. Section 4 introduces the hybridization matrix, in which each class is associated with a specific frequency sequence, and establishes a relationship between this matrix and its eigenvalues. This section also presents the results concerning the general form of the fuzzy function and shows that, in the computation of the fuzzy function for polyhybridization, the formulation derived for the dihybridization case serves as a fundamental component. Finally, Section 5 provides concluding remarks and outlines potential directions for future research.

2. Preliminary Material

This section introduces the basic concepts from hyperstructure theory that will be used throughout the paper [13].

Definition 1. A hyperoperation on a nonempty set H is a map $\circ : H \times H \rightarrow \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H .

For any subsets A, B of H , define the set $A \circ B = \bigcup_{a \in A, b \in B} a \circ b$ and for each $h \in H$ we write $h \circ A$ and $A \circ h$ for $\{h\} \circ A$.

Definition 2. The pair $(H, \circ)$ is called a hypergroup, if it satisfies the associativity condition: for all a, b, c of H we have $(a \circ b) \circ c = a \circ (b \circ c)$ and reproductive law $a \circ H = H \circ a = H$.

Definition 3. The pair $(H, \circ)$ is called a semihypergroup if only the associativity condition is satisfied.

Definition 4. A commutative hypergroup $(H, \circ)$ is a join space if the following implication holds if for all a, b, c, d, x of H $a \in b \circ x, c \in d \circ x \Rightarrow a \circ d \cap b \circ c \neq \emptyset$.

The Algorithm to Determine the Fuzzy Degree on a Hypergroup

The central idea of the article is to introduce the fuzzy function in the context of genetics and to compute the fuzzy degree in the case of dihybridizations and trihybridizations. Therefore, we present the algorithm for determining the fuzzy degree of a hypergroupoid. Let $(H, \circ)$ be a hypergroupoid. To construct the associated fuzzy degree, we define the following quantities for every $z \in H$.

Step 1: Computing $\alpha_1(z)$

$$\alpha_1(z) = \sum_{z \in x \circ y} \frac{1}{|x \circ y|},$$

where $\alpha_1(z)$ quantifies the weighted occurrence of z in all hyperoperations $x \circ y$ that contain it.

Step 2: Defining the support set $Q_1(z)$

$$Q_1(z) = \{(x, y) \in H^2 \mid z \in x \circ y\}, \quad (1)$$

where $Q_1(z)$ is the set of all pairs (x, y) for which the element z appears in the hyperproduct $x \circ y$.

Step 3: Counting generating pairs

$$q_1(z) = |Q_1(z)|. \quad (2)$$

Step 4: Defining the first fuzzy function

$$\mu_1(z) = \frac{\alpha_1(z)}{q_1(z)}, \quad (3)$$

where $\mu_1(z)$ is obtained by averaging the weighted contributions of all generating pairs (x, y) that contain z in the hyperproduct $x \circ y$.

Step 5: Constructing the join space H_1 .

To the fuzzy set μ_1 , we associate a join space $H_1 = H_{\mu_1}$, where for all $(x, y) \in H^2$,

$$x \circ_1 y = \{z \mid \min\{\mu_1(x), \mu_1(y)\} \leq \mu_1(z) \leq \max\{\mu_1(x), \mu_1(y)\}\}.$$

Step 6: Recursion.

Continuing the same procedure on $H_1 = (H, \circ_1)$, we obtain $\alpha_2, Q_2, q_2, \mu_2$, and then a new join space $H_2 = (H, \circ_2)$, and so on.

The fuzzy grade ∂H is $\min \{n \mid H_n \simeq H_{n+1}\}$. If the hypergroupoid H is finite, this process produces a finite sequence [13].

3. Results on the Fuzzy Function Associated with Mendelian Inheritance

3.1. The Fuzzy Function Associated with the Dihybrid Cross

To provide a clearer perspective on the problem under study, we begin this section with a brief overview of several pertinent results presented in [6]. The primary objective here is to determine the fuzzy degree associated with the hypergroup generated by the set of phenotypes within the framework of dihybridization. Specifically, we analyze a cross between a homozygous dominant parent, $A_1A_1A_2A_2$ and a homozygous recessive parent, $a_1a_1a_2a_2$. The outcome of this hypothetical experiment can be described as follows:

$$\begin{aligned} P &: A_1A_1A_2A_2 \otimes_2 a_1a_1a_2a_2 \\ F_1 &: A_1a_1A_2a_2 \text{ and} \\ F_1 \otimes F_1 &: A_1a_1A_2a_2 \otimes_2 A_1a_1A_2a_2 \\ F_2 &: \hat{A}_1^2, \hat{A}_2^2, \hat{A}_3^2, \hat{A}_4^2 \text{ phenotypes} \end{aligned}$$

where

$$\begin{aligned} \hat{A}_1^2 &= \{A_1x_1A_2x_2 \mid x_i = A_i \text{ or } x_i = a_i, i \in \{1, 2\}\}; \\ \hat{A}_2^2 &= \{A_1x_1a_2a_2 \mid x_1 = A_1 \text{ or } x_1 = a_1\}; \\ \hat{A}_3^2 &= \{a_1a_1A_2x_2 \mid x_2 = A_2 \text{ or } x_2 = a_2\}; \\ \hat{A}_4^2 &= \{a_1a_1a_2a_2\}. \end{aligned}$$

Let $H^2 = \{\hat{A}_1^2, \hat{A}_2^2, \hat{A}_3^2, \hat{A}_4^2\}$ be the set of phenotypes. It was shown in [2] that $(H^2, \otimes)$ forms a hypergroup. The hypergroup of dihybrids is the following:

$$\mathcal{K}_{22}^2 = \begin{pmatrix} H^2 & H^2 & H^2 & H^2 \\ H^2 & K_2^2 & H^2 & K_2^2 \\ H^2 & H^2 & K_3^2 & K_3^2 \\ H^2 & K_2^2 & K_3^2 & K_4^2 \end{pmatrix}. \quad (4)$$

where $K_2^2 = \{\hat{A}_2^2, \hat{A}_4^2\}$, $K_3^2 = \{\hat{A}_3^2, \hat{A}_4^2\}$, $K_4^2 = \{\hat{A}_4^2\}$.

Proposition 1. *In the case of dihybridization, the distribution of phenotypes satisfies the following relationship:*

$$q_1^2 < q_2^2 = q_3^2 < q_4^2,$$

where $q_i^2(\hat{A}_i^2) = q_i^2$, $i \in \{1, 2, 3, 4\}$ represents the number of occurrences of the class $\hat{A}_i^2$.

Proof. Using the dihybridization matrix (4), we observe that the class $\hat{A}_1^2$ belongs only to the set H^2 , which implies $q_1^2 = 9 = 3^2$. Class $\hat{A}_2^2$ appears in the sets H^2 and K_2^2 , which means that the number of occurrences of class $\hat{A}_2^2$ is equal to $q_2^2 = q_1^2 + 3 = 12$. Similarly $\hat{A}_3^2 \in H^2$ and $\hat{A}_3^2 \in K_2^2$, so $q_3^2 = q_1^2 + 3 = 12$, and $\hat{A}_4^2$ occurs in all sets, so $q_4^2 = |H^2|^2 = 16$. Hence, the order of phenotypes is given by

$$q_1^2 < q_2^2 = q_3^2 < q_4^2.$$

□

In the following, we propose to calculate the fuzzy function associated with each class.

Proposition 2. Let $\mu : H \rightarrow [0, 1]$ be a fuzzy function associated with classes $\hat{A}_i^2$, then the distribution of phenotypes preserves the same order as the fuzzy function.

$$\mu_1(\hat{A}_1^2) < \mu_1(\hat{A}_2^2) = \mu(\hat{A}_3^2) < \mu_1(\hat{A}_4^2). \quad (5)$$

Proof. Using the algorithm presented in the previous section, we have

$$\begin{aligned} Q_1(\hat{A}_1^2) &= \left\{ (\hat{A}_i^2, \hat{A}_j^2) \in H^2 \times H^2 \mid \hat{A}_1^2 \in \hat{A}_i^2 \otimes \hat{A}_j^2, i, j \in \{1, 2, 3, 4\} \right\} = \\ &= \left\{ (\hat{A}_1^2, \hat{A}_i^2) \in H^2 \times H^2, i \in \{1, 2, 3, 4\} \right\} \cup \\ &\cup \left\{ (\hat{A}_i^2, \hat{A}_1^2) \in H^2 \times H^2, i \in \{2, 3, 4\} \right\} \cup (\hat{A}_2^2, \hat{A}_3^2) \cup (\hat{A}_3^2, \hat{A}_2^2). \end{aligned}$$

It can be observed that $|Q_1(\hat{A}_1^2)| = q_1^2 = 4 + 3 + 1 + 1 = 9$, and

$$\begin{aligned} \alpha_1(\hat{A}_1^2) &= \sum_{\hat{A}_1^2 \in \hat{A}_i^2 \otimes \hat{A}_j^2} \frac{1}{|\hat{A}_i^2 \otimes \hat{A}_j^2|} = \frac{1}{|\hat{A}_1^2 \otimes \hat{A}_1^2|} + 2 \left(\frac{1}{|\hat{A}_1^2 \otimes \hat{A}_2^2|} + \frac{1}{|\hat{A}_1^2 \otimes \hat{A}_3^2|} + \frac{1}{|\hat{A}_1^2 \otimes \hat{A}_4^2|} \right) \\ &+ 2 \frac{1}{|\hat{A}_2^2 \otimes \hat{A}_3^2|} \\ \alpha_1(\hat{A}_1^2) &= \frac{1}{4} + 2 \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) + 2 \cdot \frac{1}{4} = \frac{9}{4}. \end{aligned}$$

For $\hat{A}_2^2$, we proceed similarly

$$\begin{aligned} Q_1(\hat{A}_2^2) &= \left\{ (\hat{A}_i^2, \hat{A}_j^2) \in H^2 \times H^2 \mid \hat{A}_2^2 \in \hat{A}_i^2 \otimes \hat{A}_j^2, i, j \in \{1, 2, 3, 4\} \right\} = \\ &= \left\{ (\hat{A}_1^2, \hat{A}_i^2) \in H^2 \times H^2, i \in \{1, 2, 3, 4\} \right\} \cup \\ &\cup \left\{ (\hat{A}_i^2, \hat{A}_2^2) \in H^2 \times H^2, i \in \{2, 3, 4\} \right\} \cup (\hat{A}_2^2, \hat{A}_2^2) \cup (\hat{A}_2^2, \hat{A}_4^2) \cup (\hat{A}_4^2, \hat{A}_2^2). \end{aligned}$$

We have $q_2^2 = |Q_1(\hat{A}_2^2)| = 9 + 3 = 12$.

$$\begin{aligned} \alpha_1(\hat{A}_2^2) &= \sum_{\hat{A}_2^2 \in \hat{A}_i^2 \otimes \hat{A}_j^2} \frac{1}{|\hat{A}_i^2 \otimes \hat{A}_j^2|} = \frac{1}{|\hat{A}_1^2 \otimes \hat{A}_2^2|} + 2 \left(\frac{1}{|\hat{A}_1^2 \otimes \hat{A}_2^2|} + \frac{1}{|\hat{A}_1^2 \otimes \hat{A}_3^2|} + \frac{1}{|\hat{A}_1^2 \otimes \hat{A}_4^2|} \right) \\ &+ 2 \frac{1}{|\hat{A}_2^2 \otimes \hat{A}_3^2|} + \frac{1}{|\hat{A}_2^2 \otimes \hat{A}_2^2|} + 2 \frac{1}{|\hat{A}_2^2 \otimes \hat{A}_4^2|}. \\ \alpha_1(\hat{A}_2^2) &= \frac{9}{4} + \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{15}{4}. \end{aligned}$$

The behavior of the classes $\hat{A}_2^2$ and $\hat{A}_3^2$ is identical; therefore, we have

$$\begin{aligned} Q_1(\hat{A}_3^2) &= \left\{ (\hat{A}_i^2, \hat{A}_j^2) \in H^2 \times H^2 \mid \hat{A}_3^2 \in \hat{A}_i^2 \otimes \hat{A}_j^2, i, j \in \{1, 2, 3, 4\} \right\} = \\ &= \left\{ (\hat{A}_1^2, \hat{A}_i^2) \in H^2 \times H^2, i \in \{1, 2, 3, 4\} \right\} \cup \\ &\cup \left\{ (\hat{A}_i^2, \hat{A}_3^2) \in H^2 \times H^2, i \in \{2, 3, 4\} \right\} \cup (\hat{A}_3^2, \hat{A}_3^2) \cup (\hat{A}_3^2, \hat{A}_4^2) \cup (\hat{A}_4^2, \hat{A}_3^2). \end{aligned}$$

So, $|Q_1(\hat{A}_3^2)| = q_3^2 = 12$ and $\alpha_1(\hat{A}_3^2) = \frac{15}{4}$. The class $\hat{A}_4^2$ belongs to all sets, with $|Q_1(\hat{A}_4^2)| = q_4^2 = 16$, and $\alpha_1(\hat{A}_4^2) = \frac{9}{4} + \frac{3}{2} + \frac{3}{2} + 1 = \frac{25}{4}$. The fuzzy function associated with the class $\hat{A}_i^2$ represents the ratio between $\alpha_1(\hat{A}_i^2)$ and q_i^2 . In conclusion,

$$\begin{aligned}
\mu_1(\hat{A}_1^2) &= \frac{\alpha_1(\hat{A}_1^2)}{q_1^2} = \frac{9}{4} \cdot \frac{1}{9} = \frac{1}{4} = \frac{1}{2^2}; \\
\mu_1(\hat{A}_2^2) &= \frac{\alpha_1(\hat{A}_2^2)}{q_2^2} = \frac{15}{4} \cdot \frac{1}{12} = \frac{5}{16} = \frac{5}{2^4}; \\
\mu_1(\hat{A}_3^2) &= \frac{\alpha_1(\hat{A}_3^2)}{q_3^2} = \frac{15}{4} \cdot \frac{1}{12} = \frac{5}{16} = \frac{5}{2^4}; \\
\mu_1(\hat{A}_4^2) &= \frac{\alpha_1(\hat{A}_4^2)}{q_4^2} = \frac{25}{4} \cdot \frac{1}{16} = \frac{25}{64} = \frac{5^2}{2^6}.
\end{aligned}$$

We get the next order

$$\mu_1(\hat{A}_1^2) < \mu_1(\hat{A}_2^2) = \mu_1(\hat{A}_3^2) < \mu_1(\hat{A}_4^2),$$

which concludes the proof. $\square$

In the following, we calculate the fuzzy degrees associated with the dihybrid cross.

Theorem 1. The fuzzy degree associated with the hypergroup $H^2 = \{\hat{A}_1^2, \hat{A}_2^2, \hat{A}_3^2, \hat{A}_4^2\}$ is equal to 3.

Proof. We have, $H_1 = (H^2, \circ_1)$, where

$$x \circ_1 y = \{z / \min\{\mu_1(x), \mu_1(y)\} \leq \mu_1(z) \leq \max\{\mu_1(x), \mu_1(y)\}\}.$$

According to relation (5), we get Table 1.

Table 1. The hypergroup H_1 .

$\circ_1$	$\hat{A}_1^2$	$\hat{A}_2^2$	$\hat{A}_3^2$	$\hat{A}_4^2$
$\hat{A}_1^2$	$\hat{A}_1^2$	$\{\hat{A}_1^2, \hat{A}_2^2, \hat{A}_3^2\}$	$\{\hat{A}_1^2, \hat{A}_2^2, \hat{A}_3^2\}$	H^2
$\hat{A}_2^2$		$\{\hat{A}_2^2, \hat{A}_3^2\}$	$\{\hat{A}_2^2, \hat{A}_3^2\}$	$\{\hat{A}_2^2, \hat{A}_3^2, \hat{A}_4^2\}$
$\hat{A}_3^2$			$\{\hat{A}_2^2, \hat{A}_3^2\}$	$\{\hat{A}_2^2, \hat{A}_3^2, \hat{A}_4^2\}$
$\hat{A}_4^2$				$\hat{A}_4^2$

By applying the fuzzy function to the classes from Table 1, we obtain

$$\begin{aligned}
\mu_2(\hat{A}_1^2) &= \frac{17}{6} \cdot \frac{1}{7} = \frac{17}{42}, \quad \mu_2(\hat{A}_2^2) = \frac{31}{6} \cdot \frac{1}{14} = \frac{31}{84} = \mu_2(\hat{A}_3^2), \quad \mu_2(\hat{A}_4^2) = \frac{17}{6} \cdot \frac{1}{7} = \frac{17}{42}. \\
\mu_2(\hat{A}_2^2) &= \mu_2(\hat{A}_3^2) < \mu_2(\hat{A}_1^2) = \mu_2(\hat{A}_4^2).
\end{aligned}$$

It is necessary to continue the process and obtain $H_2 = (H^2, \circ_2)$, where

$$x \circ_2 y = \{z / \min\{\mu_2(x), \mu_2(y)\} \leq \mu_2(z) \leq \max\{\mu_2(x), \mu_2(y)\}\}.$$

Thus, the fuzzy function calculated for the elements from Table 2 is

$$\mu_3(\hat{A}_1^2) = \mu_3(\hat{A}_2^2) = \mu_3(\hat{A}_3^2) = \mu_3(\hat{A}_4^2) = \frac{1}{3}.$$

In conclusion $(H^2, \circ_3) \simeq (H^2, \circ_4)$, so $\partial H^2 = 3$.

Table 2. The hypergroup H_2 .

$\circ_2$	$\hat{A}_1^2$	$\hat{A}_2^2$	$\hat{A}_3^2$	$\hat{A}_4^2$
$\hat{A}_1^2$	$\{\hat{A}_1^2, \hat{A}_4^2\}$	H^2	H^2	$\{\hat{A}_1^2, \hat{A}_4^2\}$
$\hat{A}_2^2$		$\{\hat{A}_2^2, \hat{A}_3^2\}$	$\{\hat{A}_2^2, \hat{A}_3^2\}$	H^2
$\hat{A}_3^2$			$\{\hat{A}_2^2, \hat{A}_3^2\}$	H^2
$\hat{A}_4^2$				$\{\hat{A}_1^2, \hat{A}_4^2\}$

3.2. Fuzzy Function Associated with Trihybrid Cross Case

The article [6] analyzed the distribution of phenotypes in the case of trihybridizations. We recall some of these results to support the study of the fuzzy function on the trihybridization classes.

$$\begin{aligned}
 P &: A_1 A_1 A_2 A_2 A_3 A_3 \otimes_3 a_1 a_1 a_2 a_2 a_3 a_3 \\
 F_1 &: A_1 a_1 A_2 a_2 A_3 a_3 \text{ and} \\
 F_1 \otimes F_1 &: A_1 a_1 A_2 a_2 A_3 a_3 \otimes_3 A_1 a_1 A_2 a_2 A_3 a_3 \\
 F_2 &: \hat{A}_1^3, \hat{A}_2^3, \hat{A}_3^3, \hat{A}_4^3, \dots, \hat{A}_8^3 \text{ phenotypes}
 \end{aligned}$$

Let $H^3 = \{\hat{A}_1^3, \hat{A}_2^3, \dots, \hat{A}_8^3\}$ be the set of phenotypes, where

$$\begin{aligned}
 \hat{A}_1^3 &= \{A_1 x_1 A_2 x_2 A_3 x_3 \mid x_i = A_i \text{ or } x_i = a_i, i \in \{1, 2, 3\}\}; \\
 \hat{A}_2^3 &= \{A_1 x_1 A_2 x_2 a_3 a_3 \mid x_i = A_i \text{ or } x_i = a_i, i \in \{1, 2\}\}; \\
 \hat{A}_3^3 &= \{A_1 x_1 a_2 a_2 A_3 x_3 \mid x_i = A_i \text{ or } x_i = a_i, i \in \{1, 3\}\}; \\
 \hat{A}_4^3 &= \{A_1 x_1 a_2 a_2 a_3 a_3 \mid x_1 = A_1 \text{ or } x_1 = a_1\}; \\
 \hat{A}_5^3 &= \{a_1 a_1 A_2 x_2 A_3 x_3 \mid x_i = A_i \text{ or } x_i = a_i, i \in \{2, 3\}\}; \\
 \hat{A}_6^3 &= \{a_1 a_1 A_2 x_2 a_3 a_3 \mid x_2 = A_2 \text{ or } x_2 = a_2\}; \\
 \hat{A}_7^3 &= \{a_1 a_1 a_2 a_2 A_3 x_3 \mid x_3 = A_3 \text{ or } x_3 = a_3\}; \\
 \hat{A}_8^3 &= \{a_1 a_1 a_2 a_2 a_3 a_3\}.
 \end{aligned}$$

The Table 3 represents the relationship between phenotypes.

Table 3. The hypergroup of trihybrid cross case.

$\otimes_3$	$\hat{A}_1^3$	$\hat{A}_2^3$	$\hat{A}_3^3$	$\hat{A}_4^3$	$\hat{A}_5^3$	$\hat{A}_6^3$	$\hat{A}_7^3$	$\hat{A}_8^3$
$\hat{A}_1^3$	H^3	H^3	H^3	H^3	H^3	H^3	H^3	H^3
$\hat{A}_2^3$		K_2^3	H^3	K_2^3	H^3	K_2^3	H^3	K_2^3
$\hat{A}_3^3$			K_3^3	K_3^3	H^3	H^3	K_3^3	K_3^3
$\hat{A}_4^3$				K_4^3	H^3	K_2^3	K_3^3	K_4^3
$\hat{A}_5^3$					K_5^3	K_5^3	K_5^3	K_5^3
$\hat{A}_6^3$						K_6^3	K_5^3	K_6^3
$\hat{A}_7^3$							K_7^3	K_8^3
$\hat{A}_8^3$								K_8^3

Where $K_2^3 = \{\hat{A}_2^3, \hat{A}_4^3, \hat{A}_6^3, \hat{A}_8^3\}$, $K_3^3 = \{\hat{A}_3^3, \hat{A}_4^3, \hat{A}_7^3, \hat{A}_8^3\}$, $K_4^3 = \{\hat{A}_4^3, \hat{A}_8^3\}$, $K_5^3 = \{\hat{A}_5^3, \hat{A}_6^3, \hat{A}_7^3, \hat{A}_8^3\}$, $K_6^3 = \{\hat{A}_6^3, \hat{A}_8^3\}$, $K_7^3 = \{\hat{A}_7^3, \hat{A}_8^3\}$, $K_8^3 = \{\hat{A}_8^3\}$. The above proposition was here. The composition between phenotypes is commutative, that is, $\hat{A}_i^3 \otimes_3 \hat{A}_j^3 = \hat{A}_j^3 \otimes_3 \hat{A}_i^3$, for any $i, j \in \{1, 2, \dots, 8\}$. Using the structure $\mathcal{K}_4^2$, we observe that this same pattern repeats three times, allowing us to rewrite the previous table as follows:

$$\mathcal{K}_8^3 : \begin{pmatrix} \mathcal{K}_4^3 & \mathcal{K}_4^3 \\ \mathcal{K}_4^3 & \mathcal{K}_3^3 \end{pmatrix}$$

where $\mathcal{K}_4^3$ is

$$\mathcal{K}_4^3 = \begin{pmatrix} H^3 & H^3 & H^3 & H^3 \\ H^3 & K_2^3 & H^3 & K_2^3 \\ H^3 & H^3 & K_3^3 & K_3^3 \\ H^3 & K_2^3 & K_3^3 & K_4^3 \end{pmatrix}, \quad (6)$$

and the matrix k_3^4 is

$$k_3^4 = \begin{pmatrix} K_5^3 & K_5^3 & K_5^3 & K_5^3 \\ & K_6^3 & K_5^3 & K_6^3 \\ & & K_7^3 & K_7^3 \\ & & & K_8^3 \end{pmatrix}$$

In the following, we recall the order of phenotypes in the distribution of trihybrids.

Proposition 3. *In the case of trihybridization, the distribution of phenotypes satisfies the following relationship:*

$$q_1^3 < q_2^3 = q_3^3 = q_5^3 < q_4^3 = q_6^3 = q_7^3 < q_8^3,$$

where $q_i(\hat{A}_i^3)$ represents the number of occurrences of the phenotype $\hat{A}_i^3$.

Proof. In what follows, we denote the number of occurrences of the phenotype $\hat{A}_i^3$ by $q_i(\hat{A}_i^3)$. The first four classes $\hat{A}_1^3$, $\hat{A}_2^3$, $\hat{A}_3^3$, and $\hat{A}_4^3$ appear in the matrix $\mathcal{K}_4^3$ which shares the same properties as the matrix $\mathcal{K}_4^2$. Therefore,

$$\begin{aligned} q_1^3 &= 3 \cdot q_1^2 = 3 \cdot 3^2 = 27, \\ q_2^3 &= 3 \cdot q_2^2 = 3 \cdot 12 = 36, \\ q_3^3 &= 3 \cdot q_3^2 = 3 \cdot 12 = 36, \\ q_4^3 &= 3 \cdot q_4^2 = 3 \cdot 16 = 48. \end{aligned}$$

The classes $\hat{A}_5^3$, $\hat{A}_6^3$, $\hat{A}_7^3$, and $\hat{A}_8^3$ appear in both the matrix $\mathcal{K}_4^3$, and the matrix k_3^4 . The class $\hat{A}_5^3$ belongs to the sets H^3 and K_5^3 . Similarly, the class $\hat{A}_6^3$ belongs to the set H^3 , K_2^3 , and K_6^3 . In the same manner, we observe that the class $\hat{A}_7^3$ belongs to the sets H^3 , K_3^3 , and K_7^3 . The class $\hat{A}_8^3$ appears all the sets. Therefore, we can either count the number of occurrences for each class, or determine a relationship between the number of phenotypes given by the first four classes. We have the following connections:

$$\begin{aligned} q_5^3 &= q_1^3 + \frac{q_1^3}{3} = 36 \\ q_6^3 &= q_2^3 + \frac{q_2^3}{3} = 48 \\ q_7^3 &= q_3^3 + \frac{q_3^3}{3} = 48 \\ q_8^3 &= q_4^3 + \frac{q_4^3}{3} = 64. \end{aligned}$$

Notice that the distribution of phenotypes in the matrix k_3^4 confirms the same order as the distribution of phenotypes in $\mathcal{K}_4^3$. Therefore, we have

$$q_1^3 < q_2^3 = q_3^3 = q_5^3 < q_4^3 = q_6^3 = q_7^3 < q_8^3. \quad (7)$$

□

The calculation of the fuzzy function reveals the number of phenotype occurrences. This is the reason why we will analyze the relationship between the fuzzy function and the phenotype distribution.

Proposition 4. Let $\mu_1 : H \rightarrow [0, 1]$ be the fuzzy function associated with the classes $\hat{A}_i^3$, $i \in \{1, 2, \dots, 8\}$, then the distribution of phenotypes keeps the same order with the fuzzy function.

$$\mu_1(\hat{A}_1^3) < \mu_1(\hat{A}_2^3) = \mu_1(\hat{A}_3^3) = \mu_1(\hat{A}_5^3) < \mu_1(\hat{A}_4^3) = \mu_1(\hat{A}_6^3) = \mu_1(\hat{A}_7^3) < \mu_1(\hat{A}_8^3). \quad (8)$$

Proof. Let $\mu_1 : H \rightarrow [0, 1]$ be the fuzzy function defined by [13]. So,

$$\begin{aligned} Q_1(\hat{A}_1^3) &= \left\{ (\hat{A}_i^3, \hat{A}_j^3) \in H^3 \times H^3 \mid \hat{A}_1^3 \in \hat{A}_i^3 \otimes \hat{A}_j^3, i, j \in \{1, 2, \dots, 8\} \right\} \\ &= \left\{ (\hat{A}_1^3, \hat{A}_i^3), i \in \{1, 2, \dots, 8\} \right\} \cup \left\{ (\hat{A}_2^3, \hat{A}_i^3), i \in \{1, 3, 5, 7\} \right\} \\ &\quad \cup \left\{ (\hat{A}_3^3, \hat{A}_i^3), i \in \{1, 2, 3, 4\} \right\} \cup \left\{ (\hat{A}_4^3, \hat{A}_i^3), i \in \{1, 5\} \right\} \\ &\quad \cup \left\{ (\hat{A}_5^3, \hat{A}_i^3), i \in \{1, 2, 3, 4\} \right\} \cup \left\{ (\hat{A}_6^3, \hat{A}_i^3), i \in \{1, 3\} \right\} \\ &\quad \cup \left\{ (\hat{A}_7^3, \hat{A}_i^3), i \in \{1, 2\} \right\} \cup (\hat{A}_8^3, \hat{A}_1^3). \end{aligned}$$

Therefore, the cardinality of the set $Q_1(\hat{A}_1^3)$ represents the number of occurrences of the phenotype $\hat{A}_1^3$. $q_1^3 = |Q_1(\hat{A}_1^3)| = 27$, and

$$\begin{aligned} \alpha_1(\hat{A}_1^3) &= \sum_{\hat{A}_i^3 \in \hat{A}_1^3 \otimes \hat{A}_j^3} \frac{1}{|\hat{A}_i^3 \otimes \hat{A}_j^3|} = \frac{27}{8}. \\ \mu_1(\hat{A}_1^3) &= \frac{\alpha_1(\hat{A}_1^3)}{q_1^3} = \frac{1}{8} = \frac{1}{2} \mu_1(\hat{A}_1^2). \end{aligned}$$

Similarly, we will calculate the fuzzy function for the other phenotypes and obtain

$$\begin{aligned} \mu_1(\hat{A}_2^3) &= \frac{\alpha_1(\hat{A}_2^3)}{q_2^3} = \frac{\sum_{\hat{A}_i^3 \in \hat{A}_2^3 \otimes \hat{A}_j^3} \frac{1}{|\hat{A}_i^3 \otimes \hat{A}_j^3|}}{q_2^3} = \frac{45}{36} = \frac{5}{32} = \frac{1}{2} \mu(\hat{A}_2^2), \\ \mu_1(\hat{A}_3^3) &= \frac{\alpha_1(\hat{A}_3^3)}{q_3^3} = \frac{\sum_{\hat{A}_i^3 \in \hat{A}_3^3 \otimes \hat{A}_j^3} \frac{1}{|\hat{A}_i^3 \otimes \hat{A}_j^3|}}{q_3^3} = \frac{5}{32} = \frac{1}{2} \mu(\hat{A}_3^2), \\ \mu_1(\hat{A}_4^3) &= \frac{\alpha_1(\hat{A}_4^3)}{q_4^3} = \frac{\sum_{\hat{A}_i^3 \in \hat{A}_4^3 \otimes \hat{A}_j^3} \frac{1}{|\hat{A}_i^3 \otimes \hat{A}_j^3|}}{q_4^3} = \frac{25}{128} = \frac{1}{2} \mu(\hat{A}_4^2), \\ \mu_1(\hat{A}_5^3) &= \frac{\alpha_1(\hat{A}_5^3)}{q_5^3} = \frac{\sum_{\hat{A}_i^3 \in \hat{A}_5^3 \otimes \hat{A}_j^3} \frac{1}{|\hat{A}_i^3 \otimes \hat{A}_j^3|}}{q_5^3} = \frac{5}{32} = \mu(\hat{A}_1^3) + \frac{\mu(\hat{A}_1^3)}{4} = \frac{5}{4} \mu(\hat{A}_1^3), \\ \mu_1(\hat{A}_6^3) &= \frac{\alpha_1(\hat{A}_6^3)}{q_6^3} = \frac{\sum_{\hat{A}_i^3 \in \hat{A}_6^3 \otimes \hat{A}_j^3} \frac{1}{|\hat{A}_i^3 \otimes \hat{A}_j^3|}}{q_6^3} = \frac{25}{128} = \mu(\hat{A}_2^3) + \frac{\mu(\hat{A}_2^3)}{4} = \frac{5}{4} \mu(\hat{A}_2^3), \\ \mu_1(\hat{A}_7^3) &= \frac{\alpha_1(\hat{A}_7^3)}{q_7^3} = \frac{\sum_{\hat{A}_i^3 \in \hat{A}_7^3 \otimes \hat{A}_j^3} \frac{1}{|\hat{A}_i^3 \otimes \hat{A}_j^3|}}{q_7^3} = \frac{25}{128} = \mu(\hat{A}_3^3) + \frac{\mu(\hat{A}_3^3)}{4} = \frac{5}{4} \mu(\hat{A}_3^3), \\ \mu_1(\hat{A}_8^3) &= \frac{\alpha_1(\hat{A}_8^3)}{q_8^3} = \frac{\sum_{\hat{A}_i^3 \in \hat{A}_8^3 \otimes \hat{A}_j^3} \frac{1}{|\hat{A}_i^3 \otimes \hat{A}_j^3|}}{q_8^3} = \frac{125}{512} = \mu(\hat{A}_4^3) + \frac{\mu(\hat{A}_4^3)}{4} = \frac{5}{4} \mu(\hat{A}_4^3). \end{aligned}$$

$$\mu_1(\hat{A}_1^3) < \mu_1(\hat{A}_2^3) = \mu_1(\hat{A}_3^3) = \mu_1(\hat{A}_5^3) < \mu_1(\hat{A}_4^3) = \mu_1(\hat{A}_6^3) = \mu_1(\hat{A}_7^3) < \mu_1(\hat{A}_8^3). \quad (9)$$

□

Remark 1. It can be observed that the fuzzy function behaves similarly to the phenotype distribution.

Therefore, for ease of writing, we introduce the following equivalence relation “ $\sim$ ” on H^3 . We define “ $\sim$ ” be the relation $x \sim y$ if and only if $\mu_1(x) = \mu_1(y)$. It is obvious that, “ $\sim$ ” is an equivalence relation. So, in the case of trihybridization, we have $[A_1^3] = \hat{A}_1^3$, $[\hat{A}_2^3] = \{\hat{A}_2^3, \hat{A}_3^3, \hat{A}_5^3\}$, $[\hat{A}_4^3] = \{\hat{A}_4^3, \hat{A}_6^3, \hat{A}_7^3\}$, $[\hat{A}_8^3] = \hat{A}_8^3$. It is denoted by $[A_i^3] = x_i^3$, $i \in \{1, 2, 4, 8\}$.

Theorem 2. The fuzzy degree associated with the hypergroup $H^3 = \{\hat{A}_1^3, \hat{A}_2^3, \dots, \hat{A}_8^3\}$ is equal to 2.

Proof. According to the inequalities given by relation (9), we can state that $(H^3, \circ_1)$ has Table 4:

Table 4. The hypergroup H_1 .

$\circ_1$	$\hat{A}_1^3$	$\hat{A}_2^3$	$\hat{A}_3^3$	$\hat{A}_4^3$	$\hat{A}_5^3$	$\hat{A}_6^3$	$\hat{A}_7^3$	$\hat{A}_8^3$
$\hat{A}_1^3$	x_1^3	x_1^3, x_2^3	x_1^3, x_2^3	x_1^3, x_2^3, x_4^3	x_1^3, x_2^3	x_1^3, x_2^3, x_4^3	x_1^3, x_2^3, x_4^3	H^3
$\hat{A}_2^3$		x_2^3	x_2^3	x_2^3, x_4^3	x_2^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3, x_8^3
$\hat{A}_3^3$			x_2^3	x_2^3, x_4^3	x_2^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3, x_8^3
$\hat{A}_4^3$				x_4^3	x_2^3, x_4^3	x_4^3	x_4^3	x_4^3, x_8^3
$\hat{A}_5^3$					x_2^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3, x_8^3
$\hat{A}_6^3$						x_4^3	x_4^3	x_4^3, x_8^3
$\hat{A}_7^3$							x_4^3	x_4^3, x_8^3
$\hat{A}_8^3$								x_8^3

By analyzing the fuzzy function for the elements in Table 4, we obtain

$$\begin{aligned} \mu_2(\hat{A}_1^3) &= \left(1 + 2 \cdot \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{7} + \frac{1}{4} + \frac{1}{7} + \frac{1}{8}\right)\right) \cdot \frac{1}{15} = \frac{101}{420} \simeq 0.24; \\ \mu_2(\hat{A}_2^3) &= \mu_2(\hat{A}_3^3) = \mu_2(\hat{A}_5^3) = \frac{265}{28} \cdot \frac{1}{47} \simeq 0.216. \\ \mu_2(\hat{A}_4^3) &= \mu_2(\hat{A}_6^3) = \mu_2(\hat{A}_7^3) = \frac{265}{28} \cdot \frac{1}{47} \simeq 0.204 \\ \mu_2(\hat{A}_8^3) &= \frac{101}{28} \cdot \frac{1}{15} \simeq 0.24. \end{aligned}$$

The resulting order is as follows:

$$\mu_2(\hat{A}_2^3) = \mu_2(\hat{A}_3^3) = \mu_2(\hat{A}_4^3) = \mu_2(\hat{A}_5^3) = \mu_2(\hat{A}_6^3) = \mu_2(\hat{A}_7^3) < \mu_2(\hat{A}_1^3) = \mu_2(\hat{A}_8^3). \quad (10)$$

We get $(H, \circ_2)$ thus Table 5.

Table 5. The hypergroup H_2 .

$\circ_2$	$\hat{A}_1^3$	$\hat{A}_2^3$	$\hat{A}_3^3$	$\hat{A}_4^3$	$\hat{A}_5^3$	$\hat{A}_6^3$	$\hat{A}_7^3$	$\hat{A}_8^3$
$\hat{A}_1^3$	x_1^3, x_8^3	H^3	H^3	H^3	H^3	H^3	H^3	x_1^3, x_8^3
$\hat{A}_2^3$		x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	H^3
$\hat{A}_3^3$			x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	H^3
$\hat{A}_4^3$				x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	H^3
$\hat{A}_5^3$					x_2^3, x_4^3	x_2^3, x_4^3	x_2^3, x_4^3	H^3
$\hat{A}_6^3$						x_2^3, x_4^3	x_2^3, x_4^3	H^3
$\hat{A}_7^3$							x_2^3, x_4^3	H^3
$\hat{A}_8^3$								x_1^3, x_8^3

□

For the elements from Table 5, the fuzzy function associated to them is

$$\begin{aligned}\mu_3(\hat{A}_1^3) &= \mu_3(\hat{A}_8^3) = \frac{5}{28} \simeq 0.17 \\ \mu_3(\hat{A}_2^3) &= \mu_3(\hat{A}_3^3) = \mu_3(\hat{A}_4^3) = \mu_3(\hat{A}_5^3) = \mu_3(\hat{A}_6^3) = \mu_3(\hat{A}_7^3) = \frac{3}{20} \simeq 0.15.\end{aligned}$$

Therefore, the order is

$$\mu_3(\hat{A}_2^3) = \mu_3(\hat{A}_3^3) = \mu_3(\hat{A}_5^3) = \mu_3(\hat{A}_4^3) = \mu_3(\hat{A}_6^3) = \mu_3(\hat{A}_7^3) < \mu_3(\hat{A}_1^3) = \mu_3(\hat{A}_8^3).$$

In conclusion, the same order is observed as in the case of the hypergroup $(H, \circ_2)$. We can therefore state that $(H^3, \circ_2) \simeq (H^3, \circ_3)$, which implies $\partial H^3 = 2$. $\square$

4. Generalization of k Hibridization

The Connection Between Hybridization and Matrix Blocks

In the following, we aim to establish a connection between the fuzzy functions of hybridizations, based on the observations presented in the previous section. It is necessary to recall some results obtained in the article [6]. We consider dominant and recessive homozygous parents

$$\begin{aligned}P &: A_1 A_1 A_2 A_2 \dots A_k A_k \otimes_k a_1 a_1 a_2 a_2 \dots a_k a_k \\ F_1 &: A_1 a_1 A_2 a_2 \dots A_k a_k \text{ and} \\ F_1 \otimes F_1 &: A_1 a_1 A_2 a_2 \dots A_k a_k \otimes_k A_1 a_1 A_2 a_2 \dots A_k a_k \\ F_2 &: \hat{A}_1^k, \hat{A}_2^k, \dots, \hat{A}_{2^k}^k \text{ phenotypes.}\end{aligned}$$

Theorem 3. *In the case of k hybridization, the distribution of phenotypes is the following one*

$$q_i^k(\hat{A}_i^k) = 3q_i^{k-1}(\hat{A}_i^{k-1}), 1 \leq i \leq 2^{k-1}, \quad (11)$$

$$q_{(2^{k-1}+i)}^k(\hat{A}_{2^{k-1}+i}^k) = \frac{4}{3}q_i^k(\hat{A}_i^k), 1 \leq i \leq 2^{k-1}. \quad (12)$$

where $q_i^k(\hat{A}_i^k)$ represents the number of occurrences of the phenotype $\hat{A}_i^k$ for hybridization k .

In the article [6] is noticed that the phenotypes $\hat{A}_i^k$ and $\hat{A}_{2^{k-1}+i}^k$ occur in the same set, for any $i \in \{1, 2, \dots, 2^{k-1}\}$.

Definition 5. *The number of occurrences of the class $\hat{A}_i^k$ in the set K_j^k is denoted by $\gamma_{i,j}^k$, where $j \in \{1, 2, \dots, 2^k\}$, $k \geq 2$ is a natural number*

The function $f: H^k \rightarrow \mathbb{R}^{2^k}$, is considered, where

$$f(\hat{A}_i^k) = (\gamma_{i,1}^k, \gamma_{i,2}^k, \dots, \gamma_{i,2^k}^k), \quad (13)$$

with the property as $\sum_{j=1}^{2^k} \gamma_{i,j}^k = q_i^k$, $i \in \{1, 2, \dots, 2^k\}$, k is a natural number, $k \geq 2$.

Remark 2. *In the case of dihybridization, the function $f: H^2 \rightarrow \mathbb{R}^{2^2}$,*

$$\begin{aligned}f(\hat{A}_1^2) &= (3^2, 0, 0, 0), f(\hat{A}_2^2) = (3^2, 3, 0, 0) \\ f(\hat{A}_3^2) &= (3^2, 0, 3, 0), f(\hat{A}_4^2) = (3^2, 3, 3, 1).\end{aligned}$$

In the case of trihybrids, we have $f : H^3 \rightarrow \mathbb{R}^{2^3}$,

$$\begin{aligned} f(\hat{A}_1^3) &= (3^3, 0, 0, 0, 0, 0, 0, 0), & f(\hat{A}_2^3) &= (3^3, 3^2, 0, 0, 0, 0, 0, 0) \\ f(\hat{A}_3^3) &= (3^3, 0, 3^2, 0, 0, 0, 0, 0), & f(\hat{A}_4^3) &= (3^3, 3^2, 3^2, 3, 0, 0, 0, 0) \\ f(\hat{A}_5^3) &= (3^3, 0, 0, 0, 3^2, 0, 0, 0), & f(\hat{A}_6^3) &= (3^3, 3^2, 0, 0, 3^2, 3, 0, 0) \\ f(\hat{A}_7^3) &= (3^3, 0, 3^2, 0, 3^2, 0, 3, 0), & f(\hat{A}_8^3) &= (3^3, 3^2, 3^2, 3, 3^2, 3, 3, 1). \end{aligned}$$

Notation 1. For each k -hybridization, we denote by M_{2^k} the matrix associated with the hybridization, consisting of $f(\hat{A}_i^k)$, $i \in \{1, 2, \dots, 2^k\}$.

Proposition 5. The matrix relationship between dihybridizations and trihybridizations is:

$$M_{2^3} = \begin{pmatrix} 3M_{2^2} & O_{2^2} \\ 3M_{2^2} & M_{2^2} \end{pmatrix}.$$

Proof. In the case of dihybridizations and trihybridizations, we have the matrices.

$$M_{2^2} = \begin{pmatrix} 3^2 & 0 & 0 & 0 \\ 3^2 & 3 & 0 & 0 \\ 3^2 & 0 & 3 & 0 \\ 3^2 & 3 & 3 & 1 \end{pmatrix}, \quad M_{2^3} = \begin{pmatrix} 3^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3^3 & 3^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3^3 & 0 & 3^2 & 0 & 0 & 0 & 0 & 0 \\ 3^3 & 3^2 & 3^2 & 3 & 0 & 0 & 0 & 0 \\ 3^3 & 0 & 0 & 0 & 3^2 & 0 & 0 & 0 \\ 3^3 & 3^2 & 0 & 0 & 3^2 & 3 & 0 & 0 \\ 3^3 & 0 & 3^2 & 0 & 3^2 & 0 & 3 & 0 \\ 3^3 & 3^2 & 3^2 & 3 & 3^2 & 3 & 3 & 1 \end{pmatrix}.$$

Hence, the conclusion is obvious. $\square$

Theorem 4.

$$\lambda(M_{2^k}) = \{1, 3, 3^3, \dots, 3^k\},$$

$k \geq 2$ is a natural number and $\lambda(M_{2^k})$ represents the set of eigenvalues associated with the matrix M_{2^k} .

Proof. The inductive method does the proof. Therefore, we denote by

$$P(k) : \lambda(M_{2^k}) = \{1, 3, 3^3, \dots, 3^k\}, \quad (14)$$

k is a natural number, $k \geq 2$. For $k = 2$, we have the matrix M_{2^2} diagonal, which implies that the eigenvalues of the matrix are $\{1, 3, 3^2\}$, i.e., the elements on the main diagonal. We assume that $P(k)$ is true and want to show that $P(k+1)$ is true, which implies

$$\lambda(M_{2^{k+1}}) = \{1, 3, 3^3, \dots, 3^k, 3^{k+1}\}.$$

Since $M_{2^k} = \begin{pmatrix} 3M_{2^{k-1}} & O_{2^{k-1}} \\ 3M_{2^{k-1}} & M_{2^{k-1}} \end{pmatrix}$, we obtain $M_{2^{k+1}} = \begin{pmatrix} 3M_{2^k} & O_{2^k} \\ 3M_{2^k} & M_{2^k} \end{pmatrix}$. Taking into account the form of the block matrix, we can state that $\text{Spec}(M_{2^{k+1}}) = \lambda(3M_{2^k}) \cup \lambda(M_{2^k}) = \{1, 3, 3^2, \dots, 3^k, 3^{k+1}\}$. Therefore $P(k+1)$ is true, which implies $P(k)$ true for any k natural number, $k \geq 2$. $\square$

Proposition 6. The relationship between two consecutive hybridizations is as follows:

$$f(\hat{A}_i^k) = (3\gamma_{i,1}^{k-1}, 3\gamma_{i,2}^{k-1}, \dots, 3\gamma_{i,2^{k-1}}^{k-1}, 0, 0, \dots, 0), \quad (15)$$

$$f(\hat{A}_{2^{k-1}+i}^k) = (3\gamma_{i,1}^{k-1}, 3\gamma_{i,2}^{k-1}, \dots, 3\gamma_{i,2^{k-1}}^{k-1}, \gamma_{i,2}^{k-1}, \dots, \gamma_{i,2^{k-1}}^{k-1}), \quad (16)$$

$$i \in \{1, 2, \dots, 2^{k-1}\}.$$

Proof. In the demonstration, the table describing the distribution of phenotypes presented in [6] is used, having the following form

$$\mathcal{K}_{2^k}^k : \begin{pmatrix} \mathcal{K}_{2^{k-1}}^k & \mathcal{K}_{2^{k-1}}^{k-1} \\ \mathcal{K}_k^k & k_{2^{k-1}}^{k-1} \end{pmatrix},$$

where $\mathcal{K}_{2^{k-1}}^k$ represents the description of the phenotype distribution for $k-1$ hybridization, and $k_{2^{k-1}}^{k-1} = \mathcal{K}_{2^{k-1}}^{k-1} \setminus \mathcal{K}_{2^{k-1}}^k$. Classes of the type $\hat{A}_i^k$, $i \in \{1, 2, \dots, 2^{k-1}\}$ are observed to appear only in the $\mathcal{K}_{2^{k-1}}^k$ structures and do not appear in the $k_{2^{k-1}}^{k-1}$ structure, which implies the writing $f(\hat{A}_i^k) = (3\gamma_{i,1}^{k-1}, 3\gamma_{i,2}^{k-1}, \dots, 3\gamma_{i,2^{k-1}}^{k-1}, 0, 0, \dots, 0)$. In the case of classes of type $\hat{A}_{2^{k-1}+i}^k$, $i \in \{1, 2, \dots, 2^{k-1}\}$ these are found in both structures, what the given form implies (16). $\square$

Remark 3. The relationship between the cardinalities of the sets is

$$|K_j^k| = 2|K_j^{k-1}|, j \in \{1, 2, \dots, 2^{k-1} - 1\} \quad (17)$$

$$|K_{2^{k-1}+j}^k| = \frac{1}{2}|K_j^k|, j \in \{1, 2, \dots, 2^{k-1} - 1\}. \quad (18)$$

$$|K_{2^{k-1}}^k| = 2. \quad (19)$$

Theorem 5. The connection relating the fuzzy functions associated with two consecutive hybridizations is given by:

$$\mu_1(\hat{A}_i^k) = \frac{1}{2}\mu_1(\hat{A}_i^{k-1}), i \in \{1, 2, \dots, 2^{k-1}\}, \quad (20)$$

$$\mu_1(\hat{A}_{2^{k-1}+i}^k) = \frac{5}{4}\mu_1(\hat{A}_i^k), i \in \{1, 2, \dots, 2^{k-1}\}. \quad (21)$$

Proof. The definition of the fuzzy function is used, according to the algorithm presented in the previous section, thus

$$\begin{aligned} \mu_1(\hat{A}_i^k) &= \frac{\alpha_1(\hat{A}_i^k)}{q(\hat{A}_i^k)}, \quad q_1(\hat{A}_i^k) = |Q(\hat{A}_i^k)|, \\ Q_1(\hat{A}_i^k) &= \{(\hat{A}_j^k, \hat{A}_i^k) \in H^k \mid \hat{A}_i^k \in \hat{A}_j^k \otimes \hat{A}_t^k\}, \\ \alpha_1(\hat{A}_i^k) &= \sum_{(\hat{A}_j^k, \hat{A}_t^k) \in Q(\hat{A}_i^k)} \frac{1}{|\hat{A}_j^k \otimes \hat{A}_t^k|}, \quad j, t \in \{1, 2, \dots, 2^k\}. \end{aligned}$$

So,

$$\mu_1(\hat{A}_i^k) = \frac{1}{q_i^k} \left(\sum_{j=1}^{2^k} \frac{\gamma_{i,j}^k}{|K_j^k|} \right) \stackrel{(15)}{=} \frac{1}{3q_i^{k-1}} \cdot \frac{3}{2} \left(\sum_{j=1}^{2^{k-1}} \frac{\gamma_{i,j}^{k-1}}{|K_j^{k-1}|} \right) = \frac{1}{2q_i^{k-1}} \left(\sum_{j=1}^{2^{k-1}} \frac{\gamma_{i,j}^{k-1}}{|K_j^{k-1}|} \right) = \frac{1}{2}\mu_1(\hat{A}_i^{k-1}),$$

$i \in \{1, 2, \dots, 2^{k-1}\}$, where $K_1^k = H^k$. It has been proven that the classes $\hat{A}_i^k$ and $\hat{A}_{2^{k-1}+i}^k$ are in the same sets. We use the form given by (16) and obtain

$$\begin{aligned} \mu_1(\hat{A}_{2^{k-1}+i}^k) &= \frac{1}{q_{2^{k-1}+i}^k} \left(\sum_{j=1}^{2^k} \frac{\gamma_{2^{k-1}+i,j}^k}{|K_j^k|} \right) \stackrel{(16)}{=} \frac{3}{4q_i^k} \cdot \left(\sum_{j=1}^{2^{k-1}} \frac{\gamma_{i,j}^k}{|K_j^k|} \right) + \frac{3}{4q_i^k} \cdot \left(\sum_{j=1}^{2^{k-1}} \frac{\gamma_{i,j}^{k-1}}{|K_{2^{k-1}+j}^k|} \right) \\ &\stackrel{(18)}{=} \frac{3}{4q_i^k} \cdot \left(\sum_{j=1}^{2^{k-1}} \frac{\gamma_{i,j}^k}{|K_j^k|} \right) + \frac{6}{4 \cdot 3 \cdot q_i^{k-1}} \cdot \left(\sum_{j=1}^{2^{k-1}} \frac{\gamma_{i,j}^{k-1}}{2|K_j^{k-1}|} \right) \\ &= \frac{3}{4}\mu_1(\hat{A}_i^k) + \frac{6}{8}\mu_1(\hat{A}_i^{k-1}) = \frac{3}{4}\mu_1(\hat{A}_i^k) + \frac{1}{4} \cdot \mu_1(\hat{A}_i^{k-1}) = \frac{3}{4}\mu_1(\hat{A}_i^k) + \frac{2}{4}\mu_1(\hat{A}_i^k) \\ &= \frac{5}{4}\mu_1(\hat{A}_i^k) = \frac{5}{8}\mu_1(\hat{A}_i^{k-1}). \end{aligned}$$

Which concludes the demonstration. $\square$

The explicit writing of the fuzzy function associated with a class $\hat{A}_i^k$, can be explained as follows:

Theorem 6. Let hybridization k be, then the fuzzy function can be rewritten as follows: for $1 \leq i \leq 2^{k-1}$, $\mu_1(\hat{A}_i^k) = \frac{1}{2^{k-p}}\mu_1(\hat{A}_i^p)$ and similarly, $\mu_1(\hat{A}_{2^{k-1}+i}^k) = \frac{5}{4}\mu_1(\hat{A}_i^k) = \frac{5}{2^{p+2}}\mu_1(\hat{A}_i^p)$, where $\mu_1(\hat{A}_i^p)$ represents the initial value of the associated fuzzy function.

Proof. The proof follows immediately from the theorem presented previously. $\square$

Example 1. Let $k \geq 4$ and consider the classes $\hat{A}_i^k, i \in \{1, 2, \dots, 8\}$. Then,

$$\mu_1(\hat{A}_1^k) = \frac{1}{2}\mu_1(\hat{A}_1^{k-1}) = \frac{1}{2^2}\mu_1(\hat{A}_1^{k-2}) = \dots = \frac{1}{2^{k-2}}\mu_1(\hat{A}_1^2) = \frac{1}{2^{k-2}} \cdot \frac{1}{2^2} = \frac{1}{2^k}, k \geq 2.$$

Similarly, let the class $\hat{A}_2^k$

$$\mu_1(\hat{A}_2^k) = \frac{1}{2}\mu_1(\hat{A}_2^{k-1}) = \frac{1}{2^2}\mu_1(\hat{A}_2^{k-2}) = \dots = \frac{1}{2^{k-2}}\mu_1(\hat{A}_2^2) = \frac{1}{2^{k-2}} \cdot \frac{5}{2^4} = \frac{5}{2^{k+2}}, k \geq 2.$$

Because $\mu_1(\hat{A}_2^2) = \mu_1(\hat{A}_3^2)$, implies $\mu_1(\hat{A}_2^k) = \mu_1(\hat{A}_3^k)$.

$$\begin{aligned} \mu_1(\hat{A}_4^k) &= \frac{1}{2^{k-2}}\mu_1(\hat{A}_4^2) = \frac{1}{2^{k-2}} \cdot \frac{5^2}{2^6} = \frac{5^2}{2^{k+4}}, \\ \mu_1(\hat{A}_5^k) &= \frac{1}{2^{k-3}}\mu_1(\hat{A}_5^3) = \frac{1}{2^{k-3}} \cdot \frac{5}{4} \cdot \mu_1(\hat{A}_1^3) = \frac{5}{2^{k-1}} \cdot \frac{1}{2} \cdot \mu_1(\hat{A}_1^2) = \frac{5}{2^{k+2}}. \end{aligned}$$

Hence, $\mu_1(\hat{A}_5^k) = \mu_1(\hat{A}_2^k) = \mu_1(\hat{A}_3^k)$.

$$\begin{aligned} \mu_1(\hat{A}_6^k) &= \frac{1}{2^{k-3}}\mu_1(\hat{A}_6^3) = \frac{1}{2^{k-3}} \cdot \frac{5}{4} \cdot \mu_1(\hat{A}_2^3) = \frac{5}{2^{k-1}} \cdot \frac{1}{2} \cdot \mu_1(\hat{A}_2^2) = \frac{5}{2^{k-1}} \cdot \frac{1}{2} \cdot \frac{5}{2^4} = \frac{5^2}{2^{k+4}} \\ \mu_1(\hat{A}_6^k) &= \mu_1(\hat{A}_7^k), \text{ because } \mu_1(\hat{A}_2^2) = \mu_1(\hat{A}_3^2). \\ \mu_1(\hat{A}_8^k) &= \frac{1}{2^{k-3}}\mu_1(\hat{A}_8^3) = \frac{1}{2^{k-3}} \cdot \frac{5}{4} \cdot \mu_1(\hat{A}_4^3) = \frac{1}{2^{k-3}} \cdot \frac{5}{4} \cdot \frac{1}{2} \cdot \mu_1(\hat{A}_4^2) = \frac{5}{2^k} \cdot \frac{5^2}{2^6} = \frac{5^3}{2^{k+6}}. \end{aligned}$$

It is observed that, in calculating the fuzzy function associated with a class for a k -hybridization, the classes from the dihybridization case are used. Therefore, a generalization of the fuzzy function is presented in the following result.

Proposition 7. Let the set $H^k = \{\hat{A}_1^k, \hat{A}_2^k, \dots, \hat{A}_{2^k}^k\}$, then the relationship between the fuzzy function associated with the classes $\hat{A}_i^k$, $i \in \{1, 2, \dots, 2^k\}$ and the classes $\hat{A}_1^2, \hat{A}_2^2, \hat{A}_3^2, \hat{A}_4^2$ is the following:

$$\begin{aligned}\mu_1(\hat{A}_i^k) &= \frac{5^s}{2^{k-2+2s}} \cdot \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^{p-s}}^2), 4 < i \leq 2^{k-1}; \\ \mu_1(\hat{A}_i^k) &= \frac{1}{2^{k-2}} \cdot \mu_1(\hat{A}_i^2), i \in \{1, 2, 3, 4\}. \\ \mu_1(\hat{A}_{2^{k-1}+i}^k) &= \frac{5^{s+1}}{2^{k+2s}} \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^{p-s}}^2), 4 < i \leq 2^{k-1}, \\ \mu_1(\hat{A}_{2^{k-1}+i}^k) &= \frac{5}{2^k} \cdot \mu_1(\hat{A}_i^2), i \in \{1, 2, 3, 4\}.\end{aligned}$$

where p represents the first occurrence of the class $\hat{A}_i$ and s is a natural number, such that $1 \leq i - 2^{p-1} - \dots - 2^{p-s} \leq 4$, for $i > 4$

Proof. We consider $1 \leq i \leq 2^{k-1}$ and apply the Theorem 5. The initial occurrence of the class $\hat{A}_i^p$ is denoted by $\hat{A}_i^p$.

$$\begin{aligned}\mu_1(\hat{A}_i^k) &= \frac{1}{2} \mu_1(\hat{A}_i^{k-1}) = \dots = \frac{1}{2^{k-p}} \mu_1(\hat{A}_i^p) \\ \mu_1(\hat{A}_i^p) &= \frac{5}{4} \mu_1(\hat{A}_{i-2^{p-1}}^p) = \dots = \left(\frac{5}{4}\right)^s \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^s}^p) \\ \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^s}^p) &= \frac{1}{2^{p-2}} \cdot \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^{p-s}}^2). \\ \mu_1(\hat{A}_i^k) &= \frac{1}{2^{k-p}} \cdot \left(\frac{5}{4}\right)^s \cdot \frac{1}{2^{p-2}} \cdot \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^{p-s}}^2) \\ \mu_1(\hat{A}_i^k) &= \frac{5^s}{2^{k-2+2s}} \cdot \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^s}^2)\end{aligned}$$

Also, for $i > 4$, the following situation is analyzed

$$\mu_1(\hat{A}_{2^{k-1}+i}^k) = \frac{5}{4} \mu_1(\hat{A}_i^k) = \frac{5}{4} \frac{5^s}{2^{k-2+2s}} \cdot \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^{p-s}}^2) = \frac{5^{s+1}}{2^{k+2s}} \mu_1(\hat{A}_{i-2^{p-1}-\dots-2^{p-s}}^2).$$

The same procedure follows for $i \in \{1, 2, 3, 4\}$, which concludes the proof. $\square$

In the following, a connection between the number of phenotypes associated with a class $\hat{A}_i^k$, denoted q_i^k , $1 \leq i \leq 2^k$ and the corresponding fuzzy function.

Theorem 7. Let k -hybridization, then the fuzzy function associated of classes $\hat{A}_i^k$ is the following:

$$\begin{aligned}\mu_1(\hat{A}_i^k) &= \frac{3^k + 2(q_i^k - 3^k)}{2^k q_i^k}, 1 \leq i \leq 2^{k-1} - 1 \\ \mu_1(\hat{A}_{2^{k-1}}^k) &= \left(\frac{3^k + 2(q_{2^{k-1}}^k - 3^k - 3) + 3 \cdot 2^{k-1}}{2^k} \right) \cdot \frac{1}{q_{2^{k-1}}^k}, i = 2^{k-1} \\ \mu_1(\hat{A}_{2^{k-1}+i}^k) &= \frac{5 \cdot 3^k + 10(q_i^k - 3^k)}{2^{k+2} q_i^k}, 1 \leq i \leq 2^{k-1} - 1, \\ \mu_1(\hat{A}_{2^k}^k) &= \frac{5 \cdot 3^k + 10(q_{2^{k-1}}^k - 3^k - 3) + 15 \cdot 2^{k-1}}{2^{k+2} q_{2^{k-1}}^k}.\end{aligned}$$

Proof. For $1 \leq i \leq 2^{k-1} - 1$, the classes $\hat{A}_i^k$ belong only to the sets $K_j^k, 1 \leq j \leq 2^{k-1} - 1$. According to the notations given by (13) we have

$$\mu_1(\hat{A}_i^k) = \left(\frac{\gamma_{i,1}^k}{|H^k|} + \frac{\gamma_{i,2}^k}{|K_2^k|} + \dots + \frac{\gamma_{i,2^{k-1}-1}^k}{|K_{2^{k-1}-1}^k|} \right) \cdot \frac{1}{q_i^k}.$$

Since $|H^k| = 2^k$ and using the relation $|K_j^k| = 2|K_j^{k-1}|, j \in \{1, 2, \dots, 2^{k-1} - 1\}$, we can state that $|K_j^k| = 2^{k-1}, 1 \leq j \leq 2^k - 1$, and $\gamma_{i,1}^k = 3^k, 1 \leq i \leq 2^k - 1$ according to (15). Finally, we obtain

$$\mu_1(\hat{A}_i^k) = \left(\frac{3^k}{2^k} + \frac{\sum_{t=1}^{2^{k-1}-1} \gamma_{i,t}^k}{2^{k-1}} \right) \cdot \frac{1}{q_i^k} = \left(\frac{3^k + 2(q_i^k - \gamma_{i,1}^k)}{2^k} \right) \cdot \frac{1}{q_i^k} = \frac{3^k + 2(q_i^k - 3^k)}{2^k q_i^k}.$$

Let $i = 2^{k-1}$, where $K_{2^{k-1}}^k = \{A_{2^{k-1}}, A_{2^k}\}$, so

$$\begin{aligned} \mu_1(\hat{A}_{2^{k-1}}^k) &= \left(\frac{\gamma_{2^{k-1},1}^k}{|H^k|} + \frac{\gamma_{2^{k-1},2}^k}{|K_2^k|} + \dots + \frac{\gamma_{2^{k-1},2^{k-1}-1}^k}{|K_{2^{k-1}-1}^k|} + \frac{\gamma_{2^{k-1},2^k}^k}{|K_{2^k}^k|} \right) \cdot \frac{1}{q_{2^{k-1}}^k} \\ &= \left(\frac{3^k}{2^k} + \frac{\sum_{t=2}^{2^{k-1}-1} \gamma_{2^{k-1},t}^k}{2^{k-1}} + \frac{3}{2} \right) \cdot \frac{1}{q_{2^{k-1}}^k} \\ &= \left(\frac{3^k + 2(q_{2^{k-1}}^k - 3^k - 3) + 3 \cdot 2^{k-1}}{2^k} \right) \cdot \frac{1}{q_{2^{k-1}}^k}. \end{aligned}$$

In the case where $i \in \{2^{k-1} + 1, \dots, 2^k\}$, the relation (21) applies, and the conclusion is immediate. $\square$

Example 2. Theorem 7 applies in the trihybrid situation. $H^3 = \{\hat{A}_1^3, \hat{A}_2^3, \dots, \hat{A}_8^3\}$, where $q_1^3 = 27, q_2^3 = q_3^3 = q_5^3 = 36, q_4^3 = q_6^3 = q_7^3 = 48, q_8^3 = 64$. In the first case, $i \in \{1, 2, 3\}$

$$\begin{aligned} \mu_1(\hat{A}_1^3) &= \frac{3^3 + 2(q_1^3 - 3^3)}{2^3 q_1^3} = \frac{27 + 2(27 - 27)}{8 \cdot 27} = \frac{1}{8}; \\ \mu_1(\hat{A}_2^3) &= \frac{3^3 + 2(q_2^3 - 3^3)}{2^3 q_2^3} = \frac{27 + 2(36 - 27)}{8 \cdot 36} = \frac{45}{8 \cdot 36} = \frac{5}{32}; \\ \mu_1(\hat{A}_3^3) &= \mu_1(\hat{A}_2^3) = \frac{5}{32}. \end{aligned}$$

For $i = 4$, the second case is analyzed

$$\begin{aligned} \mu_1(\hat{A}_4^3) &= \left(\frac{3^3 + 2(q_{2^2}^3 - 3^3 - 3) + 3 \cdot 2^{3-1}}{2^3} \right) \cdot \frac{1}{q_{2^2}^3} \\ &= \frac{27 + 2(48 - 27 - 3) + 12}{8 \cdot 48} = \frac{25}{128}. \end{aligned}$$

For $i \in \{5, 6, 7\}$, the third case is analyzed:

$$\begin{aligned}\mu_1(\hat{A}_5^3) &= \frac{5 \cdot 3^3 + 10(q_1^3 - 3^3)}{2^{3+2}q_1^3} = \frac{5 \cdot 27 + 10(27 - 27)}{2^5 \cdot 27} = \frac{5}{32}; \\ \mu_1(\hat{A}_6^3) &= \frac{5 \cdot 3^3 + 10(q_2^3 - 3^3)}{2^{3+2}q_2^3} = \frac{5 \cdot 27 + 10(36 - 27)}{2^5 \cdot 36} = \frac{25}{128}; \\ \mu_1(\hat{A}_7^3) &= \frac{5 \cdot 3^3 + 10(q_3^3 - 3^3)}{2^{3+2}q_3^k} = \mu_1(\hat{A}_6^3) = \frac{25}{128}.\end{aligned}$$

For $i = 8$, the last case is analyzed:

$$\mu_1(\hat{A}_8^3) = \frac{5 \cdot 3^3 + 10(q_{22}^3 - 3^3 - 3) + 15 \cdot 2^{3-1}}{2^{3+2}q_{22}^3} = \frac{125}{512}.$$

Theorem 8. Let $p \in \{0, 1, \dots, k\}$, then fuzzy function associated with the classes $\hat{A}_{2^p}^k$ is

$$\mu_1(\hat{A}_{2^p}^k) = \frac{5^p}{2^{k+2p}}, \quad p \in \{0, 1, \dots, k\}.$$

Proof. The relationship is demonstrated using the inductive procedure. Let $P(k)$: $\mu_1(\hat{A}_{2^p}^k) = \frac{5^p}{2^{k+2p}}$, $p \in \{0, 1, \dots, k\}$. be the proposition. The verification step is considered $k = 2, p \in \{0, 1, 2\}$ and let $p = 0$, $\mu_1(\hat{A}_1^2) = \frac{5^0}{2^{2+0}} = \frac{1}{4}$, and for $p = 1$, $\mu_1(\hat{A}_2^2) = \frac{5}{2^4}$ and $p = 2$, $\mu_1(\hat{A}_4^2) = \frac{5^2}{2^{2+4}} = \frac{25}{64}$. Assume $P(k)$ true and show that $P(k+1)$ is true. $P(k+1)$:

$$\mu_1(\hat{A}_{2^{p+1}}^{k+1}) = \frac{5^p}{2^{k+1+2p}}, \quad p \in \{0, 1, \dots, k, k+1\}.$$

We use the relation (20) and we obtain $\mu_1(\hat{A}_{2^{p+1}}^{k+1}) = \frac{1}{2}\mu_1(\hat{A}_{2^p}^k) \stackrel{P(k)}{=} \frac{1}{2} \cdot \frac{5^p}{2^{k+2p}} = \frac{5^p}{2^{k+1+2p}}$.
□

In what follows, we propose to determine an order between the fuzzy functions and the classes of the form $\hat{A}_{2^p}^k$, $p \in \{0, 1, \dots, k\}$.

Proposition 8. After k hybridizations, then $\mu_1(\hat{A}_1^k) < \mu_1(\hat{A}_2^k) < \mu_1(\hat{A}_{2^2}^k) < \dots < \mu_1(\hat{A}_{2^k}^k)$.

Proof. Let $p_1 < p_2$, the following inequalities occur $\mu_1(\hat{A}_{2^{p_1}}^k) = \frac{5^{p_1}}{2^{k+2p_1}} = \frac{5^{p_1}}{2^k \cdot 2^{2p_1}} = \frac{1}{2^k} \cdot \left(\frac{5}{4}\right)^{p_1} < \frac{1}{2^k} \cdot \left(\frac{5}{4}\right)^{p_2} = \mu_1(\hat{A}_{2^{p_2}}^k)$, hence the conclusion. □

If we consider the set H_1 formed only by the classes associated with the powers of 2, thus $H_{2^k}^k = \{\hat{A}_1^k, \hat{A}_2^k, \hat{A}_{2^2}^k, \dots, \hat{A}_{2^k}^k\}$, we obtain the hypergroup.

The hypergroup from Table 6 was studied in the paper [4].

Table 6. Hypergroup formed by classes.

$\circ_1$	a_1	a_2	a_{2^2}	$\dots$	a_{2^k}
a_1	$\{a_1\}$	$\{a_1, a_2\}$	$\{a_1, a_2, a_{2^2}\}$	$\dots$	$\{a_1, a_2, \dots, a_{2^k}\}$
a_2		$\{a_2\}$	$\{a_2, a_{2^2}\}$	$\dots$	$\{a_2, a_{2^2}, \dots, a_{2^k}\}$
a_{2^2}			$\{a_{2^2}\}$	$\dots$	$\{a_{2^2}, \dots, a_{2^k}\}$
$\vdots$				$\dots$	
a_{2^k}					$\{a_{2^k}\}$

5. Conclusions and Future Research

The combination of hyperoperations and fuzzy membership offers a structural method for quantifying the degree of support associated with each possible genetic outcome. Unlike empirical rule-based fuzzy systems, the fuzzy values in our framework arise directly from the hyperoperation, without relying on heuristic rules. This leads to a consistent algebraic mechanism for representing uncertainty in inheritance, in which the fuzzy membership of each element reflects the intrinsic behavior of the genetic operation.

This study extends the research initiated by [6], which analyzed the distribution of phenotypes in cases of simple dominance. The results presented here highlight that the number of phenotypes within a class represents an essential factor in defining the fuzzy function associated with that class. This observation has allowed for the development and analysis of fuzzy functions in cases of dihybridization, trihybridization, and polyhybridization, offering a broader mathematical framework for describing genetic variability through fuzzy logic.

In the case of dihybridizations, the order of the fuzzy function

$$\mu_1(\hat{A}_1^2) < \mu_1(\hat{A}_2^2) = \mu(\hat{A}_3^2) < \mu_1(\hat{A}_4^2), \quad (22)$$

led to the determination of the fuzzy degree of the hypergroup $H^2 = \{\hat{A}_1^2, \hat{A}_2^2, \hat{A}_3^2, \hat{A}_4^2\}$, as $\partial H^2 = 3$.

For trihybridizations, the following order was obtained:

$$\mu(\hat{A}_1^3) < \mu(\hat{A}_2^3) = \mu(\hat{A}_3^3) = \mu(\hat{A}_5^3) < \mu(\hat{A}_4^3) = \mu(\hat{A}_6^3) = \mu(\hat{A}_7^3) < \mu(\hat{A}_8^3),$$

which led to determining the fuzzy degree for the hypergroup $H^3 = \{\hat{A}_1^3, \hat{A}_2^3, \dots, \hat{A}_8^3\}$ is $\partial H^3 = 2$. A direct relationship between the fuzzy functions associated with dihybridizations and trihybridizations was also established, expressed as $\mu_1(\hat{A}_i^3) = \frac{1}{2}\mu_1(\hat{A}_i^2)$ and $\mu_1(\hat{A}_{4+i}^3) = \frac{5}{4}\mu_1(\hat{A}_i^3)$, where $i \in \{1, 2, 3, 4\}$.

These findings provide a foundation for generalizing the analysis to cases of polyhybridization. It has been demonstrated that the fuzzy function associated with any class can be determined after a fixed number of hybridizations (Theorem 6), and that it can be calculated based on the number of phenotypes in the class and the number of hybridizations (Theorem 7). Furthermore, the study confirms that the order induced by the fuzzy function coincides with the order defined by the distribution of the corresponding phenotypes (Proposition 8), as previously discussed in [6]. Additionally, to each class $\hat{A}_i^k$, with $i \in \{1, 2, \dots, 2^k\}$, a frequency sequence was associated (see Definition 5), resulting in the hybridization matrix. It is noted that the eigenvalues of the hybridization matrix coincide with the elements of the frequency sequence (see Theorem 4).

In conclusion, the fuzzy modeling of genetic phenomena provides a flexible and coherent framework for describing the gradual variation and inherent uncertainty in genetic inheritance. The mathematical connections identified between different levels of hybridization contribute to a deeper understanding of the relationship between genotype and phenotype in complex systems. Future research will aim to extend this analysis to cases of incomplete dominance and to develop computational methods for determining the fuzzy degree in a more efficient and automated manner.

While the present work focuses on establishing a theoretical framework for modeling genetic hybridization via fuzzy hypergroupoids, several directions for future research remain open. In particular, integrating the proposed algebraic structure with empirical statistical methods—such as causal inference frameworks or MCMC based optimization—could provide a bridge between structural uncertainty and data-driven estimation. Such an integration

would allow the fuzzy degree and the associated stabilization process to be evaluated on real genetic datasets, potentially linking algebraic invariants with observable biological variation.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The author would like to thank the referees for their advice, which significantly contributed to improving the article.

Conflicts of Interest: The author declares no conflicts of interest.

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