Propagation of Love waves in linear elastic isotropic Cosserat materials

Marius Apetrii[1], Emilian Bulgariu[2], Ionel-Dumitrel Ghiba[3], Hassam Khan[4] and Patrizio Neff[5]

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Abstract

We investigate the propagation of Love waves in an isotropic half-space modelled as a linear elastic isotropic Cosserat material. To this aim, we show that a method commonly used to study Rayleigh wave propagation is also applicable to the analysis of Love wave propagation. This approach is based on the explicit solution of an algebraic Riccati equation, which operates independently of the traditional Stroh formalism. The method provides a straightforward numerical algorithm to determine the wave amplitudes and speeds. Beyond its numerical simplicity, the method guarantees the existence and uniqueness of a subsonic wave speed, addressing a problem that remains unresolved in most Cosserat solids generalised continua theories. Although often overlooked, proving the existence of an admissible solution is, in fact, the key point that validates or invalidates the entire analytical approach used to derive the equation determining the wave speed. Interestingly, it is confirmed that the Love waves do not need the artificial introduction of a surface layer, as indicated in the literature.

Keywords: Cosserat elastic materials, matrix analysis method, Riccati equation, Love waves, existence and uniqueness, numerical simulations.

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Contents

[1]Marius Apetrii, Department of Mathematics, Alexandru Ioan Cuza University of Iaşi, Blvd. Carol I, no. 11, 700506 Iaşi, Romania; and Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi Branch, 700505 Iaşi, email: mapetrii@uaic.ro
[2]Emilian Bulgariu, "Ion Ionescu de la Brad" Iasi University of Life Sciences, Romania, email: emilian.bulgariu@iuls.ro
[3]Ionel-Dumitrel Ghiba, Department of Mathematics, Alexandru Ioan Cuza University of Iaşi, Blvd. Carol I, no. 11, 700506 Iaşi, Romania; and Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi Branch, 700505 Iaşi, email: dumitrel.ghiba@uaic.ro
[4]Hassam Khan, Department of Mathematics and Statistics, Institute of Business and Management Sciences (IOBM), Karachi. Pakistan, email: dr.hassam@iobm.edu.pk
[5]Patrizio Neff, Head of Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann Str. 9, 45127 Essen, Germany, email: patrizio.neff@uni-due.de

1 Introduction

Wave propagation in complex materials has been a subject of strong interest due to its significance in fields such as seismology, material science, and engineering. It is important to study the propagation of surface waves in other media, not only on Earth surface, due to their applications in tomography and in the construction and simulation of various metamaterials which are able to absorb the effect of such waves.

Among the various types of surface waves, the two types of surface waves are named Love waves [25] and Rayleigh waves [36]. There is a fundamental difference between these two type of surface waves. Love waves have only a horizontal motion that moves the surface from side to side perpendicular to the direction the wave is traveling, but not up and down, while the Rayleigh waves appear to roll like waves on an ocean, it moves the ground up and down, and forward and backward in the direction that the wave is moving. For both these two type of surface waves, it is considered that the boundary of the half-space modelling the body is free of surface traction. Beside these, it is considered that the non-vanishing displacements and stresses decay to zero with respect to the depth.

Love waves are of particular importance because of their ability to propagate along the surface of elastic media with distinct polarisation and dispersion properties. In seismology, Love waves are seismic surface waves responsible for the horizontal shifting of the Earth's surface during an earthquake and they are observed in experiments and measurements, but theoretically confirmed in the framework of clasical model of isotropic and homegeneous elastic materials only when when a low-velocity layer overlies a higher-velocity layer or sub-layers. The motion of particles in a Love wave follows a horizontal line perpendicular to the direction of propagation, making them transverse waves. The amplitude, or maximum particle motion, tends to decrease rapidly with depth but not along the Earth's surface. The slower decay rate along the Earth's surface makes surface waves, including Love waves, persist longer over distances compared to body waves, which dissipate faster as they propagate in three dimensions. Large earthquakes can produce Love waves that travel around the Earth multiple times before fading away. Due to their slow decay, Love waves are the most destructive seismic waves outside the immediate vicinity of an earthquake's focus or epicentre. These waves are often what people feel most strongly during an earthquake. However, our study is not only about the Love waves propagation along the Earth, it is rather a theoretical study which may be applied to an arbitrary medium modelled with the help of the linear Cosserat theory [5, 8, 31].

The study of Love waves in generalized continuum theories, such as Cosserat materials [5, 8, 31, 7], presents an intriguing opportunity to extend our understanding of wave behaviour beyond the classical framework of linear elasticity. Cosserat materials, also known as micropolar media, incorporate additional degrees of freedom, including rotational motions and couple stresses, allowing them to capture size effects and micro-structural influences that are absent in traditional continuum models. These distinctive features make Cosserat materials particularly suited for modelling media with internal structures, such as composites, granular materials, and metamaterials. In the framework of the Cosserat theory, the propagation of Rayleigh waves in an isotropic Cosserat elastic half space was studied in [4] using the Stroh formalism [6, 38]. The strong point of the approach in [4] is that explicit expressions of the attenuating coefficients and explicit conditions upon wave speed were found, as well as the exact expressions of three linear independent amplitude vectors. Then, a simple form of the secular equation is written, which, by comparison with other generalized forms of the secular equation for Cosserat materials [7, 18, 19, 20, 9, 16] does not involve the complex form of the attenuating coefficients. However, from [4] it is not obvious how to avoid in general the spurious roots of the secular equation, as long as the Stroh formalism is used. For Rayleigh waves it is shown in [14] that the new form of the secular equations, written in terms of the impedance matrix, does not admit any spurious root, i.e. there exists only one subsonic surface wave solution of the secular equation. Going back to the Love waves, their propagation in isotropic elastic Cosserat materials has been previously studied in [35, 17, 21, 22]. Indeed, [17, 21, 22] seem to be the first studies in which a qualitatively new wave mode is observed for the Cosserat elastic model (actually they are waves of Love-type). However, one important part of the study is yet missing, i.e., the proof that the secular equation really has an admissible solution that ensures that the strength of the Love wave decreases with depth. In our present study this fact is rigorously proved and confirms the remark made in [17, 21, 22].

To this aim we use the method introduced by Fu and Mielke [10] and Mielke and Fu [29] in classical linear elasticity. Motivated by previous work of Mielke and Sprenger [30] which is more related to control theory [15] than to surface wave propagation, Fu and Mielke [10] and Mielke and Fu [29] have devised a new method for anisotropic elastic materials which is not based on the Stroh formalism, it is conceptually different from the other methods and it is mathematically well explained. They have shown that the *impedance matrix* [13] defining the secular equation is the solution of an *algebraic Riccati equation*. Using the properties of this equation, in [29, 10] it is then shown that the secular equation does not admit spurious roots. It can be said that the works by Mielke and Fu give an elegant final answer to the general case of anisotropic materials, in the framework of classical linear elasticity. For the classical anisotropic model, the same form of this equation is present in the paper by Mielke and Sprenger [30] for $v = 0$ and in the papers by Fu and Mielke [29, 10] in the case of general v, see also the earlier work by Biryukov [2]. In [14] we have used Fu and Mielke's method [10, 29] in the study of the propagation of Rayleigh wave in linear elastic Cosserat materials. One of the reviewers' question at that time was if it is not possible to do the same complete study but for the Love wave propagation. Our expectation was that the entire approach presented by Fu and Mielke [29, 10] should be suitable in the Cosserat theory for Love wave propagation, too. We mention that while the study of Love wave propagation in classical linear isotropic elastic materials is easier than the study of Rayleigh waves since it reduce to a simple ordinary differential equation, at the first look, for Cosserat materials the difficulty of the studies of Love waves propagation and of the Rayleigh wave propagation have at least the same difficulty level. In fact, during our investigation we observed that in some points of our analysis, the Love wave propagation problem comes with some additional difficulties, here, determining the analytical form of the limiting speed. However, *we prove both the existence and the uniqueness* of the wave speed of the Love waves in Cosserat materials.

The structure of the present paper is now as follows. In Section 2 we present the system of partial differential equations which describe the behaviour of the material in the linear theory of isotropic and homogeneous Cosserat elastic materials and we give necessary and sufficient conditions for real plane waves propagation. In Section 3 we present the setup of the propagation of Love waves. In Section 4 we define and prove the existence of the limiting speed, we obtain the Riccati equation and the secular equation for the wave speed propagation, and we give our main result which proves the existence and uniqueness of the solution of the secular equation in the range fixed by the limiting speed. We mention that this final step is rarely possible in generalised models if other methods are used. In Section 5 we provide a numerical algorithms which can be implemented for any material once the constitutive coefficients are known. We present effective numerical results for dense polyurethane Foam@0.18mm with the constitutive parameters given in [24, 23]. We remark that while the propagation speed of the Rayleigh waves is slower than the propagation speed of the Love waves along the Earth's surface, in the dense polyurethane Foam@0.18mm it is vice versa. In the last section we give a comparative study with the classical elasticity theory of homogeneous isotropic materials.

2 Preliminaries

We consider that the mechanical behaviour of a body occupying the unbounded regular region of three dimensional Euclidean space is modelled with the help of the Cosserat theory of linear isotropic elastic materials. We denote by n the outward unit normal on $\partial\Omega$. The body is referred to a fixed system of rectangular Cartesian axes $Ox_i\,(i = 1, 2, 3)$, $\{e_1, e_2, e_3\}$ being the unit vectors of these axes. For further notations please see the Appendix A

2.1 The linear isotropic Cosserat model for elastic solids

In the Cosserat model for isotropic materials two vector fields are used to describe the macro- and micro-behaviour of the solid body, i.e., the classical displacement $u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$ and the microrotation vector field $\vartheta : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}^3$, $\vartheta = \operatorname{axl}\mathbf{A}$, where $\mathbf{A} = \operatorname{Anti}(\vartheta), \mathbf{A} \in \mathfrak{so}(3)$ represents the microrotation tensor in the Cosserat theory. In the framework of the linear isotropic hyperelastic theory the Cosserat model is based on the elastic energy density [33, 27, 26, 28, 32]

$$
\begin{aligned}
W(\mathrm{D}u, \mathrm{Anti}\,\vartheta, \mathrm{D}\vartheta) = & \underbrace{\mu_{\mathrm{e}}\,\|\mathrm{dev}_3\,\mathrm{sym}\,\mathbf{e}\|^2 + \mu_{\mathrm{c}}\,\|\mathrm{skew}\,\mathbf{e}\|^2 + \frac{2\,\mu_{\mathrm{e}} + 3\,\lambda_{\mathrm{e}}}{6}\,[\mathrm{tr}\,(\mathbf{e})]^2}_{:=W_1(\mathbf{e})} \\
& + \underbrace{\mu_{\mathrm{e}} L_{\mathrm{c}}^2 \left[\ \alpha_1\,\|\mathrm{dev}_3\,\mathrm{sym}\,\mathfrak{K}\|^2 + \alpha_2\,\|\mathrm{skew}\,\mathfrak{K}\|^2 + \frac{2\,\alpha_1 + 3\,\alpha_3}{6}\,[\mathrm{tr}\,(\mathfrak{K})]^2\right]}_{:=W_2(\mathfrak{K})},
\end{aligned} \tag{2.1}
$$

where we have used the following definitions for the independent constitutive variables $\mathbf{e}$ (the non-symmetric strain tensor) and $\mathfrak{K}$ (the curvature tensor)

$$
\mathbf{e} := \mathrm{D}u - \mathbf{A} = \mathrm{D}u - \mathrm{Anti}\,(\vartheta), \qquad\qquad \mathfrak{K} := \mathrm{D}\vartheta. \tag{2.2}
$$

Moreover, the stress-strain relations for the homogeneous isotropic Cosserat elastic solid are

$$
\begin{aligned}
\boldsymbol{\sigma} &:= \frac{\partial W}{\partial \mathbf{e}} = 2\,\mu_{\mathrm{e}}\,\mathrm{sym}\,\mathbf{e} + 2\,\mu_{\mathrm{c}}\,\mathrm{skew}\,\mathbf{e} + \lambda_{\mathrm{e}}\,\mathrm{tr}(\mathbf{e})\,\mathbb{1}, \\
\mathbf{m} &:= \frac{\partial W}{\partial \mathfrak{K}} = \mu_{\mathrm{e}} L_{\mathrm{c}}^2\,\left[2\,\alpha_1\,\mathrm{sym}\,\mathfrak{K} + 2\,\alpha_2\,\mathrm{skew}\,\mathfrak{K} + \alpha_3\,\mathrm{tr}(\mathfrak{K})\,\mathbb{1}\right],
\end{aligned} \tag{2.3}
$$

where $\boldsymbol{\sigma}$ is the non-symmetric force stress tensor and $\mathbf{m}$ is the second-order non-symmetric couple stress tensor.

The constants $(\mu_{\mathrm{e}}, \lambda_{\mathrm{e}})$, $\mu_{\mathrm{c}}, L_{\mathrm{c}}$ and $(\alpha_1, \alpha_2, \alpha_3)$ are the elastic moduli representing the parameters related to the meso-scale, the parameters related to the micro-scale the Cosserat couple modulus, the characteristic length, and the three general isotropic curvature parameters (weights), respectively. Due to the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{gl}(3) \cong \mathbb{R}^{3\times 3}$, the strict positive definiteness of the potential energy is equivalent to the following simple relations for the introduced parameters [33]

$$
\mu_{\mathrm{e}} > 0, \qquad \mu_{\mathrm{c}} > 0, \qquad 2\,\mu_{\mathrm{e}} + 3\,\lambda_{\mathrm{e}} > 0, \qquad \alpha_1 > 0, \qquad \alpha_2 > 0, \qquad 2\,\alpha_1 + 3\,\alpha_3 > 0. \tag{2.4}
$$

However, our entire subsequent analysis will be made under weaker conditions on the constitutive parameters, see (2.11).

In the absence of external body forces and of external body moment, the Euler-Lagrange equations constructed based on this internal energy are given as the following the PDE-system of the model [7]

$$
\begin{aligned}
\rho\, u_{,tt} &= \mathrm{Div}[2\,\mu_{\mathrm{e}}\,\mathrm{sym}\,\mathrm{D}u + 2\,\mu_{\mathrm{c}}\,\mathrm{skew}(\mathrm{D}u - \mathbf{A}) + \lambda_{\mathrm{e}}\,\mathrm{tr}(\mathrm{D}u)\cdot\mathbb{1}]\,, \\
2\,\rho\, j\, \mu_{\mathrm{e}}\ \tau_{\mathrm{c}}^2\ \ (\mathrm{axl}\ \mathbf{A})_{,tt} &= \mu_{\mathrm{e}} L_{\mathrm{c}}^2 \left[\alpha_1\,\mathrm{dev}\,\mathrm{sym}(\mathrm{D}\,\mathrm{axl}\,\mathbf{A}) + \alpha_2\,\mathrm{skew}(\mathrm{D}\,\mathrm{axl}\,\mathbf{A}) + \alpha_3\,\mathrm{tr}(\mathrm{D}\,\mathrm{axl}\,\mathbf{A})\cdot\mathbb{1}\right] \\
&\quad - 4\,\mu_{\mathrm{c}}\,\mathrm{axl}\,(\mathrm{skew}\,\mathrm{D}u - \mathbf{A}) \quad \text{in} \quad \Omega\times[0,T],
\end{aligned} \tag{2.5}
$$

which rewritten in indices read

$$
\begin{aligned}
\rho\,\frac{\partial^2\, u_i}{\partial\, t^2} &= \underbrace{(\mu_{\mathrm{e}} + \mu_{\mathrm{c}})\sum_{l=1}^{3}\frac{\partial^2 u_i}{\partial x_l^2} + (\mu_{\mathrm{e}} - \mu_{\mathrm{c}} + \lambda_{\mathrm{e}})\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_l \partial x_i} + 2\,\mu_{\mathrm{c}}\sum_{l,s=1}^{3}\varepsilon_{ils}\frac{\partial \vartheta_s}{\partial x_l}}_{=\mathrm{Div}\,\boldsymbol{\sigma}}, \\
\rho\, j\, \mu_{\mathrm{e}}\,\tau_{\mathrm{c}}^2\,\frac{\partial^2\,\vartheta_i}{\partial\, t^2} &= \underbrace{\mu_{\mathrm{e}} L_{\mathrm{c}}^2\left[(\alpha_1 + \alpha_2)\sum_{l=1}^{3}\frac{\partial^2 \vartheta_i}{\partial x_l^2} + (\alpha_1 - \alpha_2 + \alpha_3)\sum_{l=1}^{3}\frac{\partial^2 \vartheta_l}{\partial x_l \partial x_i}\right]}_{=\,\mathrm{Div}\,\mathbf{m}} + 2\,\mu_{\mathrm{c}}\sum_{l,s=1}^{3}\varepsilon_{isl}\frac{\partial u_l}{\partial x_s} - 4\,\mu_{\mathrm{c}}\,\vartheta_i,
\end{aligned} \tag{2.6}
$$

with $i = 1, 2, 3$, where ρ is the density, j is an inertia weight parameter, τ_{c} the internal characteristic time (then the combination $j\,\mu_{\mathrm{e}}\,\tau_{\mathrm{c}}^2$ describes the mass density of inertia moment) [7, page 163] and ε_{ils} is the Levi-Civita symbol. These equations are in complete agreement to the equations proposed in the Cosserat theory [7], but written in indices, see [12] for a complete comparison.

In the following we assume $\rho > 0$ and $j > 0$ without mentioning these conditions in the hypothesis of our results.

2.2 Real plane waves in the direction $\xi = (\xi_1, \xi_2, 0)^T$

Even if we will consider the propagation of surface waves with the direction $e_1 = (1,0,0)^T$, i.e. some horizontal direction, x_2 being the vertical direction orthogonal to the surface along which the wave decays, the method we use in this paper needs to consider a general direction of wave propagation $\xi = (\xi_1, \xi_2, 0)^T$ and to characterise where the wave is a real bulk wave. Therefore, for $\xi = (\xi_1, \xi_2, 0)^T$ with $\|\xi\|^2 = 1$, it is useful to know for which conditions on the constitutive parameters we have that for every wave number $k > 0$ the system of partial differential equations (2.6) admits a non trivial solution in the form

$$u(x_1,x_2,t) = \begin{pmatrix} 0 \\ 0 \\ \widehat{u}_3 \end{pmatrix} e^{\mathrm{i}\left(k\langle \xi,\, x\rangle_{\mathbb{R}^3} - \omega\, t\right)}, \qquad \vartheta(x_1,x_2,t) = \mathrm{i}\begin{pmatrix} \widehat{\vartheta}_1 \\ \widehat{\vartheta}_2 \\ 0 \end{pmatrix} e^{\mathrm{i}\left(k\langle \xi,\, x\rangle_{\mathbb{R}^3} - \omega\, t\right)}, \tag{2.7}$$
$$(\widehat{u}_3, \widehat{\vartheta}_1, \widehat{\vartheta_2})^T \in \mathbb{C}^3, \quad (\widehat{u}_3, \widehat{\vartheta}_1, \widehat{\vartheta_2})^T \neq 0$$

only for real positive values ω^2, where $\mathrm{i} = \sqrt{-1}$ is the complex unit.

However, since our formulation is isotropic, by demanding real plane waves in any direction $\xi = (\xi_1, \xi_2, 0)$, $\|\xi\| = 1$, it is equivalent to demanding real plane waves in the direction $e_1 = (1,0,0)$. Inserting (2.7) into (2.6) we see that $\widehat{u}_3, \widehat{\vartheta}_1$ and $\widehat{\vartheta}_2$ have to satisfy the system of linear algebraic equations

$$\begin{aligned}
-\omega^2 \rho\, \widehat{u}_3 &= -k^2\,(\mu_{\rm e} + \mu_{\rm c})\widehat{u}_3 - 2\,k\,\mu_{\rm c}\,\widehat{\vartheta}_2,\\
-\mathrm{i}\,\omega^2 \rho\, j\,\mu_{\rm e}\,\tau_{\rm c}^2\,\widehat{\vartheta}_1 &= -\mathrm{i}\,k^2\,\mu_{\rm e}\,L_{\rm c}^2\,(\alpha_1+\alpha_2)\,\widehat{\vartheta}_1 - \mathrm{i}\,k^2\,\mu_{\rm e}\,L_{\rm c}^2\,(\alpha_1-\alpha_2+\alpha_3)\widehat{\vartheta}_1 - 4\,\mathrm{i}\,\mu_{\rm c}\,\widehat{\vartheta}_1,\\
-\mathrm{i}\,\omega^2 \rho\, j\,\mu_{\rm e}\,\tau_{\rm c}^2\,\widehat{\vartheta}_2 &= -\mathrm{i}\,k^2\,\mu_{\rm e}\,L_{\rm c}^2\,(\alpha_1+\alpha_2)\,\widehat{\vartheta}_2 - 2\,\mathrm{i}\,k\,\mu_{\rm c}\,\widehat{u}_3 - 4\,\mathrm{i}\,\mu_{\rm c}\,\widehat{\vartheta}_2.
\end{aligned}\tag{2.8}$$

In [14] we find that for all $k > 0$ the system $(2.6)_2$ admits non trivial solutions $w = (\widehat{u}_3, \widehat{\vartheta}_1, \widehat{\vartheta}_2)^T$ only for real positive values ω^2 if and only if the following eigenvalue problem admits only real solutions

$$\left[\underbrace{\begin{pmatrix} k^2\,\frac{\mu_{\rm e}+\mu_{\rm c}}{\rho} & 0 & -2\,k\,\frac{\mu_{\rm c}}{\rho\sqrt{j\,\mu_{\rm e}\,\tau_{\rm c}^2}} \\ 0 & k^2\,\mu_{\rm e}\,L_{\rm c}^2\,\frac{2\,\alpha_1+\alpha_3}{\rho\, j\,\mu_{\rm e}\,\tau_{\rm c}^2} + 4\,\frac{\mu_{\rm c}}{\rho\, j\,\mu_{\rm e}\,\tau_{\rm c}^2} & 0 \\ -2\,k\,\frac{\mu_{\rm c}}{\rho\sqrt{j\,\mu_{\rm e}\,\tau_{\rm c}^2}} & 0 & k^2\,\mu_{\rm e}\,L_{\rm c}^2\,\frac{\alpha_1+\alpha_2}{\rho\, j\,\mu_{\rm e}\,\tau_{\rm c}^2} + 4\,\frac{\mu_{\rm c}}{\rho\, j\,\mu_{\rm e}\,\tau_{\rm c}^2} \end{pmatrix}}_{:=\widetilde{\mathbf{Q}}_2(e_1,k)} - \omega^2\mathbb{1}\right] h = 0, \tag{2.9}$$

with $h = \widehat{\mathbb{1}}^{1/2}\begin{pmatrix} \widehat{u}_3 \\ \widehat{\vartheta}_1 \\ \widehat{\vartheta}_2 \end{pmatrix}$ and $\widehat{\mathbb{1}} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho\, j\,\mu_{\rm e}\,\tau_{\rm c}^2 & 0 \\ 0 & 0 & \rho\, j\,\mu_{\rm e}\,\tau_{\rm c}^2 \end{pmatrix}$, i.e., if and only if

$$\mu_{\rm e} > 0, \qquad \mu_{\rm c} > 0, \qquad 2\,\alpha_1 + \alpha_3 > 0, \qquad (\alpha_1+\alpha_2) > 0. \tag{2.10}$$

Then, for $k > 0$ and due to the assumed isotropy, extrapolating to all directions of propagation, we have

Proposition 2.1. [14] *Let $\xi \in \mathbb{R}^3$, $\xi \neq 0$ be any direction of the form $\xi = (\xi_1, \xi_2, 0)^T$. The necessary and sufficient conditions for existence of a non trivial solution of the system of partial differential equations* (2.6) *of the form given by* (2.7) *are* [1]

$$\mu_{\rm e} > 0, \qquad \mu_{\rm c} > 0, \qquad \alpha_1 + \alpha_2 > 0 \qquad 2\alpha_1 + \alpha_3 > 0. \tag{2.11}$$

The restrictions (2.11) will be imposed for the rest of the paper. Note that these conditions do not involve the constitutive parameter $\lambda_{\rm e}$ and do not imply the existence of real waves in any directions.

[1] These conditions have physical interpretations, i.e., $\mu_{\rm e} > 0$ is the stability condition (for the existence of transverse waves at low frequencies that propagate with velocity $\sqrt{\frac{\mu_{\rm e}}{\rho}}$, $2\alpha_1 + \alpha_3 > 0$ is the condition for the existence of longitudinal rotational wave at high frequencies, $\alpha_1 + \alpha_2 > 0$ is the condition for the existence of the acoustic branch of the transversal waves at high frequencies, while $\mu_{\rm c} > 0$ means that the Cosserat and classical elastic effects are not uncoupled.

3 The setup for the propagation of Love waves

In the framework of the Love wave, we consider the region Ω to be the half space

$$\Sigma := \{(x_1, x_2, x_3)\,|\,x_1, x_3 \in \mathbb{R},\ x_2 \geq 0\}.$$

The boundary of the homogeneous and isotropic half-space is free of surface traction, i.e.,

$$\boldsymbol{\sigma}.\,n = 0, \qquad \mathbf{m}.\,n = 0 \qquad \text{for} \quad x_2 = 0. \tag{3.1}$$

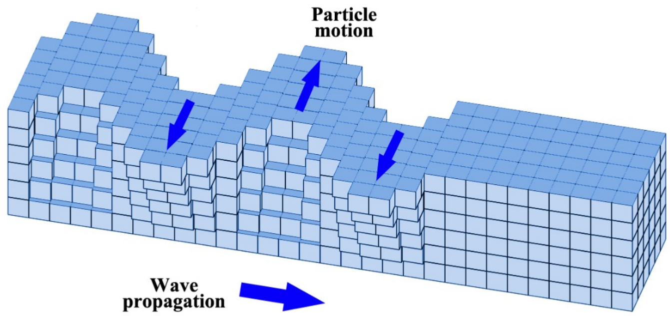


Figure 1: Love waves propagation.

In addition, the solution has to satisfy the following decay condition

$$\lim_{x_2\to\infty}\{u_3, \vartheta_1, \vartheta_2, \sigma_{13}, \sigma_{23}, m_{33}\}(x_1, x_2, t) = 0 \qquad \forall\, x_1 \in \mathbb{R}, \quad \forall\, t \in [0, \infty). \tag{3.2}$$

In isotropic solids and in the context of Love wave propagation, the surface particles move in the planes normal to the surface $x_2 = 0$ and orthogonal to the direction of propagation $e_1 = (1, 0, 0)^T$. Accordingly to these characteristics of the Love waves, we consider the following plane strain ansatz as a first step in our process of construction of the solution

$$u(x_1, x_2, t) = \begin{pmatrix} 0 \\ 0 \\ u_3(x_1, x_2, t) \end{pmatrix}, \qquad \mathrm{D}u(x_1, x_2, t) = \begin{pmatrix} 0 & 0 & \frac{\partial u_3}{\partial x_1}(x_1, x_2, t) \\ 0 & 0 & \frac{\partial u_3}{\partial x_2}(x_1, x_2, t) \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

$$\mathbf{A}(x_1, x_2, t) = \begin{pmatrix} 0 & 0 & \vartheta_2(x_1, x_2, t) \\ 0 & 0 & -\vartheta_1(x_1, x_2, t) \\ -\vartheta_2(x_1, x_2, t) & \vartheta_1(x_1, x_2, t) & 0 \end{pmatrix}, \qquad \vartheta(x_1, x_2, x_3, t) = \operatorname{axl}\mathbf{A} = \begin{pmatrix} \vartheta_1(x_1, x_2, t) \\ \vartheta_2(x_1, x_2, t) \\ 0 \end{pmatrix}.$$

Corresponding to our ansatz, we deduce the following form of the stress tensor

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & (\mu_e + \mu_c)\,\frac{\partial u_3}{\partial x_1} - 2\,\mu_c\,\vartheta_2 \\ 0 & 0 & (\mu_e + \mu_c)\,\frac{\partial u_3}{\partial x_2} + 2\,\mu_c\,\vartheta_1 \\ (\mu_e - \mu_c)\,\frac{\partial u_3}{\partial x_1} + 2\,\mu_c\,\vartheta_2 & (\mu_e - \mu_c)\,\frac{\partial u_3}{\partial x_2} - 2\,\mu_c\,\vartheta_1 & 0 \end{pmatrix}, \tag{3.4}$$

and of the couple stress tensor

$$\mathbf{m} = \mu_e\,L_c^2 \begin{pmatrix} (2\alpha_1 + \alpha_3)\,\frac{\partial \vartheta_1}{\partial x_1} + \alpha_3\,\frac{\partial \vartheta_2}{\partial x_2} & (\alpha_1 - \alpha_2)\,\frac{\partial \vartheta_1}{\partial x_2} + (\alpha_1 + \alpha_2)\,\frac{\partial \vartheta_2}{\partial x_1} & 0 \\ (\alpha_1 + \alpha_2)\,\frac{\partial \vartheta_1}{\partial x_2} + (\alpha_1 - \alpha_2)\,\frac{\partial \vartheta_2}{\partial x_1} & \alpha_3\,\frac{\partial \vartheta_1}{\partial x_1} + (2\alpha_1 + \alpha_3)\,\frac{\partial \vartheta_2}{\partial x_2} & 0 \\ 0 & 0 & \alpha_3\,\frac{\partial \vartheta_1}{\partial x_1} + \alpha_3\,\frac{\partial \vartheta_2}{\partial x_2} \end{pmatrix}, \tag{3.5}$$

while the equation of motion is reduced to

$$\begin{aligned} \rho\,\frac{\partial^2 u_3}{\partial t^2} &= \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2}, \\ \rho\,j\,\mu_e\,\tau_c^2\,\frac{\partial^2 \vartheta_1}{\partial t^2} &= \frac{\partial m_{11}}{\partial x_1} + \frac{\partial m_{21}}{\partial x_2} - \sigma_{23} + \sigma_{32}, \\ \rho\,j\,\mu_e\,\tau_c^2\,\frac{\partial^2 \vartheta_2}{\partial t^2} &= \frac{\partial m_{12}}{\partial x_1} + \frac{\partial m_{22}}{\partial x_2} + \sigma_{13} - \sigma_{31}, \end{aligned} \tag{3.6}$$

subjected to the aforementioned boundary conditions (6.9), which turn out to be

$$\sigma_{23} = 0, \qquad m_{21} = m_{22} = 0 \qquad \text{at} \qquad x_2 = 0. \tag{3.7}$$

Therefore, our further aim is to give an explicit solution $(u_3, \vartheta_1, \vartheta_2)$ of the following system

$$\begin{aligned}
\rho\, \frac{\partial^2 u_3}{\partial t^2} &= (\mu_e + \mu_c) \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} \right) - 2\,\mu_c\, \frac{\partial \vartheta_2}{\partial x_1} + 2\,\mu_c\, \frac{\partial \vartheta_1}{\partial x_2}, \qquad (3.8)\\
\rho\, j\, \mu_e\, \tau_c^2\, \frac{\partial^2 \vartheta_1}{\partial t^2} &= \mu_e\, L_c^2 \left[(2\alpha_1 + \alpha_3)\, \frac{\partial^2 \vartheta_1}{\partial x_1^2} + (\alpha_1 + \alpha_2)\, \frac{\partial^2 \vartheta_1}{\partial x_2^2} + (\alpha_1 - \alpha_2 + \alpha_3)\, \frac{\partial^2 \vartheta_2}{\partial x_1\, \partial x_2} \right] - 2\,\mu_c\, \frac{\partial u_3}{\partial x_2} - 4\,\mu_c\, \vartheta_1,\\
\rho\, j\, \mu_e\, \tau_c^2\, \frac{\partial^2 \vartheta_2}{\partial t^2} &= \mu_e\, L_c^2 \left[(\alpha_1 + \alpha_2)\, \frac{\partial^2 \vartheta_2}{\partial x_1^2} + (2\alpha_1 + \alpha_3)\, \frac{\partial^2 \vartheta_2}{\partial x_2^2} + (\alpha_1 - \alpha_2 + \alpha_3)\, \frac{\partial^2 \vartheta_1}{\partial x_1\, \partial x_2} \right] + 2\,\mu_c\, \frac{\partial u_3}{\partial x_1} - 4\,\mu_c\, \vartheta_2,
\end{aligned}$$

which satisfies the boundary conditions at $x_2 = 0$

$$\begin{aligned}
(\mu_e + \mu_c)\, \frac{\partial u_3}{\partial x_2} + 2\,\mu_c\, \vartheta_1 &= 0,\\
\mu_e\, L_c^2\, (\alpha_1 + \alpha_2)\, \frac{\partial \vartheta_1}{\partial x_2} + \mu_e\, L_c^2\, (\alpha_1 - \alpha_2)\, \frac{\partial \vartheta_2}{\partial x_1} &= 0, \qquad (3.9)\\
\mu_e\, L_c^2\, \alpha_3\, \frac{\partial \vartheta_1}{\partial x_1} + \mu_e\, L_c^2\, (2\alpha_1 + \alpha_3)\, \frac{\partial \vartheta_2}{\partial x_2} &= 0,
\end{aligned}$$

and which has the asymptotic behaviour (6.10).

4 The ansatz for the solution and the limiting speed

We look for a solution of (3.8) and (3.9) having the form[2]

$$\mathcal{U}(x_1, x_2, t) = \begin{pmatrix} u_3(x_1, x_2, t) \\ \vartheta_1(x_1, x_2, t) \\ \vartheta_2(x_1, x_2, t) \end{pmatrix} = \mathrm{Re} \left[\begin{pmatrix} \mathrm{i}\, z_3(x_2) \\ z_1(x_2) \\ z_2(x_2) \end{pmatrix} e^{\mathrm{i}\, k\, (x_1 - v t)} \right], \tag{4.1}$$

where v is the propagation speed (the phase velocity). If z_i, $i = 1, 2, 3$, are solutions of the following systems

$$\begin{gathered}
\begin{pmatrix} \mu_e + \mu_c & 0 & 0 \\ 0 & \mu_e\, L_c^2\, (\alpha_1 + \alpha_2) & 0 \\ 0 & 0 & \mu_e\, L_c^2\, (2\alpha_1 + \alpha_3) \end{pmatrix} \begin{pmatrix} z_3''(x_2) \\ z_1''(x_2) \\ z_2''(x_2) \end{pmatrix} + \mathrm{i} \begin{pmatrix} 0 & -2\mu_c & 0 \\ -2\mu_c & 0 & \mu_e\, L_c^2\, k\, (\alpha_1 - \alpha_2 + \alpha_3) \\ 0 & \mu_e\, L_c^2\, k\, (\alpha_1 - \alpha_2 + \alpha_3) & 0 \end{pmatrix} \begin{pmatrix} z_3'(x_2) \\ z_1'(x_2) \\ z_2'(x_2) \end{pmatrix}\\
- \begin{pmatrix} k^2\, (\mu_e + \mu_c) - \rho\, k^2 v^2 & 0 & 2\mu_c\, k \\ 0 & \mu_e\, L_c^2\, k^2\, (2\alpha_1 + \alpha_3) - \rho j\, \mu_e\, \tau_c^2\, k^2 v^2 + 4\mu_c & 0 \\ 2\mu_c\, k & 0 & \mu_e\, L_c^2\, k^2\, (\alpha_1 + \alpha_2) - \rho j\, \mu_e\, \tau_c^2\, k^2 v^2 + 4\mu_c \end{pmatrix} \begin{pmatrix} z_3(x_2) \\ z_1(x_2) \\ z_2(x_2) \end{pmatrix} = 0,
\end{gathered} \tag{4.2}$$

and (from the boundary conditions)

$$\begin{pmatrix} \mu_e + \mu_c & 0 & 0 \\ 0 & \mu_e\, L_c^2\, (\alpha_1 + \alpha_2) & 0 \\ 0 & 0 & \mu_e\, L_c^2\, (2\alpha_1 + \alpha_3) \end{pmatrix} \begin{pmatrix} z_3'(0) \\ z_1'(0) \\ z_2'(0) \end{pmatrix} + \mathrm{i} \begin{pmatrix} 0 & -2\,\mu_c & 0 \\ 0 & 0 & k\mu_e\, L_c^2\, (\alpha_1 - \alpha_2) \\ 0 & k\mu_e\, L_c^2\, \alpha_3 & 0 \end{pmatrix} \begin{pmatrix} z_3(0) \\ z_1(0) \\ z_2(0) \end{pmatrix} = 0,$$

where $\cdot'$ denotes the derivative with respect to x_2, then $\mathcal{U}$ given by the ansatz (4.1) satisfies (3.8) and (3.9). In a more compact notation, the above equations admit the following equivalent form

$$\begin{aligned}
\frac{1}{k^2}\, \mathbf{X}\, z''(x_2) + \mathrm{i}\, \frac{1}{k}\, (\mathbf{Y} + \mathbf{Y}^T)\, z'(x_2) - \mathbf{Z}\, z(x_2) + k^2\, v^2\, \mathbb{1}\, z(x_2) &= 0, \qquad (4.3)\\
\frac{1}{k^2}\, \mathbf{X}\, z'(0) + \mathrm{i}\, \frac{1}{k}\, \mathbf{Y}^T\, z(0) &= 0,
\end{aligned}$$

[2] We take $\mathrm{i}\, z_3$ since this choice leads us, in the end, only to real matrices.

where the matrices $\mathbf{X}\,,\mathbf{Y}\,,\mathbf{Z}$ and $\hat{\mathbb{1}}$ are defined by

$$\mathbf{X}=k^2\begin{pmatrix}\mu_{\rm e}+\mu_{\rm c} & 0 & 0\\ 0 & \mu_{\rm e}\,L_{\rm c}^2\,(\alpha_1+\alpha_2) & 0\\ 0 & 0 & \mu_{\rm e}\,L_{\rm c}^2\,(2\alpha_1+\alpha_3)\end{pmatrix},\qquad \mathbf{Y}=k\begin{pmatrix}0 & 0 & 0\\ -2\mu_{\rm c} & 0 & k\mu_{\rm e}\,L_{\rm c}^2\,\alpha_3\\ 0 & k\mu_{\rm e}\,L_{\rm c}^2\,(\alpha_1-\alpha_2) & 0\end{pmatrix},\tag{4.4}$$
$$\mathbf{Z}=\begin{pmatrix}k^2\,(\mu_{\rm e}+\mu_{\rm c}) & 0 & 2k\,\mu_{\rm c}\\ 0 & \mu_{\rm e}\,L_{\rm c}^2\,k^2\,(2\alpha_1+\alpha_3)+4\mu_{\rm c} & 0\\ 2k\,\mu_{\rm c} & 0 & \mu_{\rm e}\,L_{\rm c}^2\,k^2\,(\alpha_1+\alpha_2)+4\mu_{\rm c}\end{pmatrix},\qquad \widehat{\mathbb{1}}=\begin{pmatrix}\varrho & 0 & 0\\ 0 & \rho j\,\mu_{\rm e}\,\tau_{\rm c}^2 & 0\\ 0 & 0 & \rho j\,\mu_{\rm e}\,\tau_{\rm c}^2\end{pmatrix}.$$

The system (4.3) has a similar structure to that from classical linear elasticity [10] but we still have to rewrite it in order to make it manageable for our analysis. In this respect, the solution z of (4.3) is equivalent to find a solution y of

$$\frac{1}{k^2}\,\widehat{\mathbb{1}}^{-1/2}\,\mathbf{X}\,\widehat{\mathbb{1}}^{-1/2}\,y''(x_2)+{\rm i}\,\frac{1}{k}\,\widehat{\mathbb{1}}^{-1/2}\,(\mathbf{Y}+\mathbf{Y}^T)\widehat{\mathbb{1}}^{-1/2}\,y'(x_2)-\widehat{\mathbb{1}}^{-1/2}\,\mathbf{Z}\,\widehat{\mathbb{1}}^{-1/2}\,y(x_2)+k^2\,v^2\,\mathbb{1}\,y(x_2)=\,0,\tag{4.5}$$
$$\frac{1}{k^2}\,\widehat{\mathbb{1}}^{-1/2}\,\mathbf{X}\,\widehat{\mathbb{1}}^{-1/2}\,y'(0)+{\rm i}\,\frac{1}{k}\,\widehat{\mathbb{1}}^{-1/2}\,\mathbf{Y}^T\,\widehat{\mathbb{1}}^{-1/2}\,y(0)=\,0,$$

where $y(x_2):=\widehat{\mathbb{1}}^{1/2}\,z(x_2)$. Therefore we may use the modified matrices

$$\boldsymbol{\mathcal{X}}:=k^2\begin{pmatrix}\frac{\mu_{\rm e}+\mu_{\rm c}}{\rho} & 0 & 0\\ 0 & \frac{L_{\rm c}^2\,(\alpha_1+\alpha_2)}{\rho j\,\tau_{\rm c}^2} & 0\\ 0 & 0 & \frac{L_{\rm c}^2\,(2\alpha_1+\alpha_3)}{\rho j\,\tau_{\rm c}^2}\end{pmatrix},\qquad \boldsymbol{\mathcal{Y}}:=k\begin{pmatrix}0 & 0 & 0\\ \frac{-2\mu_{\rm c}}{\rho\,\tau_{\rm c}\sqrt{j\,\mu_{\rm e}}} & 0 & \frac{k\,L_{\rm c}^2\,\alpha_3}{\rho j\,\tau_{\rm c}^2}\\ 0 & \frac{k\,L_{\rm c}^2\,(\alpha_1-\alpha_2)}{\rho j\,\tau_{\rm c}^2} & 0\end{pmatrix},$$
$$\boldsymbol{\mathcal{Z}}:=\begin{pmatrix}\frac{k^2\,(\mu_{\rm e}+\mu_{\rm c})}{\rho} & 0 & \frac{2k\,\mu_{\rm c}}{\rho\,\tau_{\rm c}\sqrt{j\,\mu_{\rm e}}}\\ 0 & \frac{\mu_{\rm e}\,L_{\rm c}^2\,k^2\,(2\alpha_1+\alpha_3)+4\mu_{\rm c}}{\rho j\,\mu_{\rm e}\,\tau_{\rm c}^2} & 0\\ \frac{2k\,\mu_{\rm c}}{\rho\,\tau_{\rm c}\sqrt{j\,\mu_{\rm e}}} & 0 & \frac{\mu_{\rm e}\,L_{\rm c}^2\,k^2\,(\alpha_1+\alpha_2)+4\mu_{\rm c}}{\rho j\,\mu_{\rm e}\,\tau_{\rm c}^2}\end{pmatrix},\tag{4.6}$$

and the following equivalent form of the system (4.3) emerges

$$\frac{1}{k^2}\boldsymbol{\mathcal{X}}\,y''(x_2)+{\rm i}\,\frac{1}{k}(\boldsymbol{\mathcal{Y}}+\boldsymbol{\mathcal{Y}}^T)\,y'(x_2)-\boldsymbol{\mathcal{Z}}\,y(x_2)+k^2\,v^2\,\mathbb{1}\,y(x_2)=0,\tag{4.7}$$
$$\frac{1}{k^2}\boldsymbol{\mathcal{X}}\,y'(0)+{\rm i}\,\frac{1}{k}\,\boldsymbol{\mathcal{Y}}^T\,y(0)=0.$$

Lemma 4.1. *If the constitutive coefficients satisfy the conditions* (2.11)*, then the matrices* $\boldsymbol{\mathcal{Z}}$ *and* $\boldsymbol{\mathcal{X}}$ *are symmetric and positive definite.*

Proof. Symmetry is clear and it is easy to compute that

$$\mathbf{X}_{11}=k^2(\mu_{\rm e}+\mu_{\rm c})\,,\qquad \mathbf{X}_{11}\mathbf{X}_{22}-\mathbf{X}_{12}\mathbf{X}_{21}=k^4\,\mu_{\rm e}\,L_{\rm c}^2\,(\mu_{\rm e}+\mu_{\rm c})(\alpha_1+\alpha_2)\,,\tag{4.8}$$
$$\det\mathbf{X}=k^6\,\mu_{\rm e}^2\,L_{\rm c}^4\,(\mu_{\rm e}+\mu_{\rm c})(\alpha_1+\alpha_2)(2\alpha_1+\alpha_3).$$

Therefore, our constitutive hypothesis imply that $\mathbf{X}$ is positive-definite. In addition, $\mathbf{Z}$ is positive-definite if and only if the principal minors are positive, namely

$$\mathbf{Z}_{11}=k^2(\mu_{\rm e}+\mu_{\rm c}),\qquad \mathbf{Z}_{11}\mathbf{Z}_{22}-\mathbf{Z}_{12}\mathbf{Z}_{21}=k^2(\mu_{\rm e}+\mu_{\rm c})\left[\mu_{\rm e}\,L_{\rm c}^2\,k^2\,(2\alpha_1+\alpha_3)+4\mu_{\rm c}\right],\tag{4.9}$$
$$\det(\mathbf{Z})=\mu_{\rm e}\,k^2\left[\mu_{\rm e}\,L_{\rm c}^2\,k^2\,(2\alpha_1+\alpha_3)+4\mu_{\rm c}\right]\left[L_{\rm c}^2\,k^2(\mu_{\rm e}+\mu_{\rm c})(\alpha_1+\alpha_2)+4\mu_{\rm c}\right],$$

i.e. under the hypothesis of the lemma. Since $\mathbf{X}$ and $\mathbf{Z}$ are positive definite, so there are $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\mathcal{Z}}$ defined by (4.6), and the proof is complete. ■

4.1 The limiting speed

We now seek a solution y of the differential system (4.7) in the form

$$y(x_2)=\begin{pmatrix}d_3\\ d_1\\ d_2\end{pmatrix}\,e^{{\rm i}\,r\,k\,x_2},\qquad {\rm Im}\,r>0,\tag{4.10}$$

where $r \in \mathbb{C}$ is a complex parameter, $d = (d_3,\ d_1,\ d_2)^T \in \mathbb{C}^3$, $d \neq 0$ is the amplitude and $\operatorname{Im} r$ is the coefficient of the imaginary part of r. The condition $\operatorname{Im} r > 0$ ensures the asymptotic decay condition (6.10). Inserting (4.10) in $(4.7)_1$ we obtain the systems of algebraic equations

$$[r^2 \boldsymbol{\mathcal{X}} + r\,(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \boldsymbol{\mathcal{Z}} - \ k^2 v^2 \mathbb{1}]\, d = 0, \qquad [r\,\boldsymbol{\mathcal{X}} + \boldsymbol{\mathcal{Y}}^T]\, d = 0. \tag{4.11}$$

The characteristic equation corresponding to the eigenvalue problem $(4.11)_1$, i.e., the condition to have a nontrivial solution of $d = (d_3,\ d_1,\ d_2)^T$, is

$$\det\,[r^2 \boldsymbol{\mathcal{X}} + r(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \boldsymbol{\mathcal{Z}} - k^2 v^2 \mathbb{1}] = 0, \tag{4.12}$$

which gives six roots as eigenvalue r. The associated eigenvectors d can be determined for the corresponding eigenvalues.

Definition 4.2. *By the* **limiting speed** *we understand a speed $\widehat{v} > 0$, such that for all wave speeds satisfying $0 \leq v < \widehat{v}$ (subsonic speeds) the roots of the characteristic equation* (4.12) *are not real*[3] *and vice versa, i.e., if the roots of the characteristic equation* (4.12) *are not real then they correspond to wave speeds v satisfying $0 \leq v < \widehat{v}$.*

Proposition 4.3. *Let us assume that the constitutive coefficients satisfy the conditions* (2.11)*. The following holds true:*

1. *There exists a limiting speed $\widehat{v} > 0$. Furthermore, if one root r_v of the characteristic equation* (4.12) *is real then it corresponds to a speed $v \geq \widehat{v}$ (non-admissible).*
2. *For all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $k > 0$, the tensor $\boldsymbol{\mathcal{Z}}_\theta := \sin^2\theta \boldsymbol{\mathcal{X}} + \sin\theta\cos\theta(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \cos^2\theta \boldsymbol{\mathcal{Z}}$ is positive definite.*
3. *For all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $k > 0$ and $0 \leq v < \widehat{v}$, the tensor $\widetilde{\boldsymbol{\mathcal{Z}}}_\theta := \sin^2\theta\, \boldsymbol{\mathcal{X}} + \sin\theta\cos\theta(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \cos^2\theta\, \boldsymbol{\mathcal{Z}} - k^2 v^2 \cos^2\theta\, \mathbb{1}$ is positive definite.*

Proof. The proof is quite similar to the equivalent proposition concerning the Rayleigh wave, see [14]. However, since there are different calculations, we write explicitly some key points.

1. Assume that there exists a real r_v as solution of the characteristic equation (4.12), then $\exists\, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $r_v = \tan\theta$. Therefore, corresponding to (4.10), $\mathcal{U}$ given by (4.1) and defined by r_v turns into

$$\mathcal{U}(x_1, x_2, t) = \begin{pmatrix} u_3(x_1, x_2, t) \\ \vartheta_1(x_1, x_2, t) \\ \vartheta_2(x_1, x_2, t) \end{pmatrix} = \begin{pmatrix} \mathrm{i}\ d_3 \\ d_1 \\ d_2 \end{pmatrix} e^{ik\,(x_1 + \tan\theta x_2 - vt)} = \begin{pmatrix} \mathrm{i}\ d_3 \\ d_1 \\ d_2 \end{pmatrix} e^{\frac{ik}{\cos\theta}(\cos\theta x_1 + \sin\theta x_2 - \cos\theta vt)}, \tag{4.13}$$

which means that $\mathcal{U}(x_1, x_2, t)$ is a non-trivial plane body wave solution with wave number $\frac{k}{\cos\theta}$, the speed $v_\theta = v\cos\theta$ and propagation in the direction n_θ where $n_\theta = (\cos\theta, \sin\theta, 0)$. A direct substitution of (4.13) into (2.6) together with the notation $\begin{pmatrix} f_3 \\ f_1 \\ f_2 \end{pmatrix} = \widehat{\mathbb{1}}^{1/2} \begin{pmatrix} d_3 \\ d_1 \\ d_2 \end{pmatrix}$ imply the existence of a non-trivial solution $(f_1,\ f_2,\ f_3) \neq 0$ of the algebraic system written in matrix format

$$\left[\sin^2\theta\, \boldsymbol{\mathcal{X}} + \sin\theta\cos\theta(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \cos^2\theta\, \boldsymbol{\mathcal{Z}} - k^2 v_\theta^2\, \mathbb{1}\right] \begin{pmatrix} f_3 \\ f_1 \\ f_2 \end{pmatrix} = 0. \tag{4.14}$$

Let us observe that equation (4.14) is actually the propagation condition for plane waves in isotropic Cosserat materials, in the fixed direction $n_\theta = (\cos\theta, \sin\theta, 0)$. Since the constitutive coefficients satisfy

[3] Since the quadratic equation does not have real solutions, there is a complex solution r for which $\operatorname{Im} r > 0$, since the complex solutions are pair-conjugated. Therefore, the existence of such a solution r, i.e., $\operatorname{Im} r > 0$, implies the existence of a wave propagating in the direction x_1 with the phase velocity v and decaying exponentially in the direction x_2.

the conditions (2.11), according to Proposition 2.1, for the direction $n_\theta = (\cos\theta, \sin\theta, 0)$ in particular, the system of partial differential equations (2.6) admits a non trivial solution in the form

$$u(x_1,x_2,t) = \mathrm{i}\begin{pmatrix}0\\0\\\widehat{u}_3\end{pmatrix} e^{\mathrm{i}\left(k\langle\xi,\,x\rangle_{\mathbb{R}^3}-\,\omega\,t\right)}, \qquad \vartheta(x_1,x_2,t) = \begin{pmatrix}\widehat{\vartheta}_1\\\widehat{\vartheta}_2\\0\end{pmatrix} e^{\mathrm{i}\left(k\langle\xi,\,x\rangle_{\mathbb{R}^3}-\,\omega\,t\right)}, \tag{4.15}$$
$$(\widehat{u}_3, \widehat{\vartheta}_1, \widehat{\vartheta_2})^T \in \mathbb{C}^3, \quad (\widehat{u}_3, \widehat{\vartheta}_1, \widehat{\vartheta_2})^T \neq 0$$

only for real positive values ω^2. To each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we associate these real frequencies ω_θ satisfying

$$\det\{\sin^2\theta\,\boldsymbol{\mathcal{X}} + \sin\theta\cos\theta(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \cos^2\theta\,\boldsymbol{\mathcal{Z}} - \omega_\theta^2\,\mathbb{1}\} = 0. \tag{4.16}$$

Then, each ω_θ defines a v_θ such that $\omega_\theta = v_\theta\cos\theta$. We define $\widehat{v}$ as the minimum of the values of $v_\theta\cos\theta$ for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, this means that after we know all solutions v_θ of

$$\det\{\sin^2\theta\,\boldsymbol{\mathcal{X}} + \sin\theta\cos\theta(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \cos^2\theta\,\boldsymbol{\mathcal{Z}} - k^2 v_\theta^2\cos^2\theta\,\mathbb{1}\} = 0, \tag{4.17}$$

we can define

$$\widehat{v} = \inf_{\theta\in(-\frac{\pi}{2},\frac{\pi}{2})} v_\theta. \tag{4.18}$$

In conclusion, if there exists a value of v such that the equation (4.12) admits a real solution r_v, then v must satisfy $v \geq \widehat{v}$. Thus, if v is such that $0 \leq v < \widehat{v}$, then r can not be real and if r is real, then $v \geq \widehat{v}$.

2. Since the constitutive coefficients satisfy the conditions (2.11), according to Proposition 2.1, it follows that there exists *only* real numbers ω such that the following system[4] admits non trivial solutions $w = (\widehat{u}_3, \widehat{\vartheta}_1, \widehat{\vartheta}_2)^T \neq 0$ for real positive numbers ω^2, i.e.,

$$[\widetilde{\mathbf{Q}}_2(\xi,k) - \omega^2\widehat{\mathbb{1}}]\,w = 0, \tag{4.19}$$

where

$$\mathbf{Q}_2(\xi,k) = \begin{pmatrix} k^2(\mu_e+\mu_c) & -2\mu_c\,k\,\xi_2 & 2\mu_c\,k\,\xi_1\\ -2\mu_c\,k\,\xi_2 & \mu_e\,L_c^2\,k^2\left[(2\alpha_1+\alpha_3)\xi_1^2 + (\alpha_1+\alpha_2)\xi_2^2\right] + 4\mu_c & \mu_e\,L_c^2\,k^2\,(\alpha_1-\alpha_2+\alpha_3)\xi_1\xi_2\\ 2\mu_c\,k\,\xi_1 & \mu_e\,L_c^2\,k^2\,(\alpha_1-\alpha_2+\alpha_3)\xi_1\xi_2 & \mu_e\,L_c^2\,k^2\left[(\alpha_1+\alpha_2)\xi_1^2 + (2\alpha_1+\alpha_3)\xi_2^2\right] + 4\mu_c \end{pmatrix} \tag{4.20}$$

But this is equivalent to the positive definiteness of the matrix $\widetilde{\mathbf{Q}}_2(\xi,k) = \widehat{\mathbb{1}}^{-1/2}\mathbf{Q}_2(\xi,k)\widehat{\mathbb{1}}^{-1/2}$. Since the conditions (2.11) imply the positive definiteness of the matrix $\widetilde{\mathbf{Q}}_1(\xi,k)$ for all $\xi = (\xi_1, \xi_2, 0) \in \mathbb{R}^3$, $\|\xi\| = 1$, taking $\xi = (\cos\theta, \sin\theta, 0)$ we deduce the desired result.

3. Since $\boldsymbol{\mathcal{Z}}_\theta := \sin^2\theta\,\boldsymbol{\mathcal{X}} + \sin\theta\cos\theta(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \cos^2\theta\,\boldsymbol{\mathcal{Z}}$ is positive definite, it admits only positive eigenvalues. Assuming that there exist $\theta_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $v_0 \in [0, \widehat{v})$ for which there is an eigenvalue λ_{θ_0} of $\boldsymbol{\mathcal{Z}}_\theta$ such that $\lambda_{\theta_0} < k^2 v_0^2\cos^2\theta_0$, then

$$v_{\theta_0} := \sqrt{\frac{\lambda_{\theta_0}}{k^2\,\cos^2\theta_0}} < v_0 < \widehat{v} \tag{4.21}$$

[4]This system follows directly by inserting (4.15) into (2.6) and obtaining that $\widehat{u}_3, \widehat{\vartheta}_1$ and $\widehat{\vartheta}_2$ have to satisfy the following partial differential equations

$$-\omega^2\rho\,\widehat{u}_3 = -\,k^2\,(\mu_e+\mu_c)\widehat{u}_3\sum_{l=1}^{2}\xi_l^2 - 2k\mu_c\,\xi_l\,\widehat{\vartheta}_2 + 2k\mu_c\,\xi_2\,\widehat{\vartheta}_1,$$
$$\omega^2\rho\,j\,\mu_e\,\tau_c^2\,\widehat{\vartheta}_1 = k^2\,\mu_e L_c^2\,(2\alpha_1+\alpha_3)\widehat{\vartheta}_1\xi_1^2 + k^2\,\mu_e L_c^2\,(\alpha_1+\alpha_2)\widehat{\vartheta}_1\xi_2^2 + k^2\mu_e L_c^2\,(\alpha_1-\alpha_2+\alpha_3)\widehat{\vartheta}_2\xi_1\xi_2 - 2\,k\,\mu_c\widehat{u}_3\xi_2 + 4\,\mu_c\widehat{\vartheta}_1,$$
$$\omega^2\rho\,j\,\mu_e\,\tau_c^2\,\widehat{\vartheta}_2 = k^2\,\mu_e L_c^2\,(\alpha_1+\alpha_2)\widehat{\vartheta}_2\xi_1^2 + k^2\,\mu_e L_c^2\,(2\alpha_1+\alpha_3)\widehat{\vartheta}_2\xi_2^2 + k^2\mu_e L_c^2\,(\alpha_1-\alpha_2+\alpha_3)\widehat{\vartheta}_1\xi_1\xi_2 + 2\,k\,\mu_c\widehat{u}_3\xi_1 + 4\,\mu_c\widehat{\vartheta}_2.$$

is solution of (4.17), i.e., for fixed θ_0 we have that $v_{\theta_0} < \widehat{v}$ verifies

$$\det\{\sin^2\theta_0\,\boldsymbol{\mathcal{X}} + \sin\theta_0\,\cos\theta_0\,(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \cos^2\theta_0\,\boldsymbol{\mathcal{Z}} - k^2 v_{\theta_0}^2\,\cos^2\theta_0\,\mathbb{1}\} = 0. \tag{4.22}$$

This is in contradiction to the definition of the limiting speed and Proposition 4.18, since $\widehat{v}$ is the smallest speed having this property. Therefore, it remains that for all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $k > 0$ and for all $0 \leq v < \widehat{v}$, all the eigenvalues of $\boldsymbol{\mathcal{Z}}_\theta$ are larger than $k^2 v^2 \cos^2\theta$ and the proof is complete. ■

4.2 Derivation and matrix analysis of the algebraic Riccati equation

The main ingredient of the method used by Fu and Mielke [10] is to look at (4.7) as an initial value problem and to search for a solution in the form

$$y(x_2) = e^{-k\,x_2\,\boldsymbol{\mathcal{E}}}y(0), \tag{4.23}$$

where $\boldsymbol{\mathcal{E}} \in \mathbb{C}^{3\times 3}$ is to be determined. On substituting (4.23) into (4.11), we get

$$[\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{E}}^2 - \mathrm{i}\,(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T)\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{Z}} + \; k^2 v^2 \mathbb{1}]\,y(x_2) = 0, \qquad\qquad (-\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{E}} + \mathrm{i}\;\boldsymbol{\mathcal{Y}}^T)\,y(0) = 0. \tag{4.24}$$

It is clear that in order to have a proper decay the eigenvalues of $\boldsymbol{\mathcal{E}}$ have to be such that their real part is positive.

For the linear Cosserat model, we introduce the so called **surface impedance matrix**

$$\boldsymbol{\mathcal{M}} = -(-\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{E}} + \mathrm{i}\,\boldsymbol{\mathcal{Y}}^T) \qquad \iff \qquad \boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{X}}^{-1}(\boldsymbol{\mathcal{M}} + \mathrm{i}\;\boldsymbol{\mathcal{Y}}^T). \tag{4.25}$$

The reason to consider this replacement of $\boldsymbol{\mathcal{E}}$ to $\boldsymbol{\mathcal{M}}$ is to convert the equation $(4.24)_2$ into an equation for a Hermitian matrix $\boldsymbol{\mathcal{M}}$. On substituting $(4.25)_2$ into $(4.24)_1$, we obtain

$$(\boldsymbol{\mathcal{M}} - \mathrm{i}\,\boldsymbol{\mathcal{Y}})\boldsymbol{\mathcal{X}}^{-1}(\boldsymbol{\mathcal{M}} + \mathrm{i}\,\mathcal{R}^{\mathcal{T}}) - \boldsymbol{\mathcal{Z}} + k^2\,v^2\,\mathbb{1} = 0, \qquad \boldsymbol{\mathcal{M}}\,y(0) = 0. \tag{4.26}$$

We call equation $(4.26)_1$ **algebraic Riccati equation** for the propagation of Love waves in the linear Cosserat model.

From this point on, the entire problem is reduced to a purely mathematical problem. Moreover, due to the properties of the known matrices involved in the Riccati equation, as well as the definition of the limiting speed, we are exactly in the framework of the mathematical results obtained by Fu and Mielke [29]. Since we are interested in a nontrivial solution y, we impose $y(0) \neq 0$, so that the matrix $\boldsymbol{\mathcal{M}}$ has to satisfy

$$\det \boldsymbol{\mathcal{M}} = 0. \tag{4.27}$$

The equation (4.27) is called **secular equation** for the **Love waves propagation** in linear Cosserat model in terms of the **impedance matrix** $\boldsymbol{\mathcal{M}}$.

The unknown quantities are now v and $\boldsymbol{\mathcal{M}}$ solutions of the equations (4.27) and (4.26). These equations are coupled and nonlinear. On one hand, there is not only one Hermitian matrix $\boldsymbol{\mathcal{M}}$ satisfying (4.26). On the other hand, it is not clear if there exists a unique solution v of (4.27), even if all Hermitian matrix $\boldsymbol{\mathcal{M}}$ satisfying (4.26) are known. The strategy which works is to look at the equations (4.26) and to remark that equation $(4.26)_1$ leads to a mapping $v \mapsto \boldsymbol{\mathcal{M}}_v$, where $\boldsymbol{\mathcal{M}}_v$ is the solution of $(4.26)_1$ for a fixed v. Then equation (4.27) becomes the equation which determines the wave speed v. However, first we need to identify the domain of those v for which $(4.26)_1$ admits a unique solution. Then we prove the existence and uniqueness of the solution v of the secular equation. But this solution v has to be such that the corresponding solution $\boldsymbol{\mathcal{M}}_v$ of the Riccati equation (4.26) assures that $\mathrm{Re}(\mathrm{spec}\,\boldsymbol{\mathcal{E}})$ is positive, where "$\mathrm{Re}\,(\mathrm{spec}\,\boldsymbol{\mathcal{E}})$" means the "real part of spectra of $\boldsymbol{\mathcal{E}}$".

Everything we have done until now in the paper shows that we have put the problem in the framework of the general result obtained by Fu and Mielke in [11, 10] (see also the work by Mielke and Sprenger [30]) and it follows directly:

Theorem 4.4. [The main result of this paper]

1. *If* $0 \leq v < \widehat{v}$*, then the matrix problem*

$$\boldsymbol{\mathcal{X}}\mathcal{E}^2 - \mathrm{i}\,(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T)\mathcal{E} - \boldsymbol{\mathcal{Z}} + k^2 v^2 \mathbb{1} = 0, \qquad \mathrm{Re}\,\mathrm{spec}\,\boldsymbol{\mathcal{E}} > 0, \tag{4.28}$$

where $\mathrm{Re}\,\mathrm{spec}\,\boldsymbol{\mathcal{E}}$ *means the real part of the spectra of* $\boldsymbol{\mathcal{E}}$*, has a unique solution for* $\boldsymbol{\mathcal{E}}$ *and* $\boldsymbol{\mathcal{M}} := -(-\boldsymbol{\mathcal{X}}\boldsymbol{\mathcal{E}} + \mathrm{i}\,\boldsymbol{\mathcal{Y}}^T)$ *is hermitian.*

2. *If $0 \le v < \widehat{v}$, then the unique solution $\mathcal{M}$ of the algebraic Riccati equation*

$$(\boldsymbol{\mathcal{M}} - \mathrm{i}\,\boldsymbol{\mathcal{Y}})\boldsymbol{\mathcal{X}}^{-1}(\boldsymbol{\mathcal{M}} + \mathrm{i}\,\boldsymbol{\mathcal{Y}}^T) - \boldsymbol{\mathcal{Z}} + k^2\, v^2\, \mathbb{1} = 0, \qquad \boldsymbol{\mathcal{M}}\, y(0) = 0. \tag{4.29}$$

that satisfies $\mathrm{Re\,spec}\,(\boldsymbol{\mathcal{X}}^{-1}(\boldsymbol{\mathcal{M}} + \mathrm{i}\,\boldsymbol{\mathcal{Y}}^T)) > 0$ *is given explicitly by*

$$\boldsymbol{\mathcal{M}} = \Big(\int_0^\pi \boldsymbol{\mathcal{X}}_\theta^{-1}\, d\theta\Big)^{-1}\Big(\pi\mathbb{1} - \mathrm{i}\int_0^\pi \boldsymbol{\mathcal{X}}_\theta^{-1}\boldsymbol{\mathcal{Y}}_\theta^T\, d\theta\Big), \tag{4.30}$$

where the matrices $\boldsymbol{\mathcal{X}}_\theta$, $\boldsymbol{\mathcal{Y}}_\theta$ and $\boldsymbol{\mathcal{Z}}_\theta$ are obtained by rotation of the old coordinate system

$$\begin{aligned}
\boldsymbol{\mathcal{X}}_\theta &= \cos^2\theta\, \boldsymbol{\mathcal{X}} - \sin\theta\cos\theta\,(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \sin^2\theta\, \widetilde{\boldsymbol{\mathcal{Z}}},\\
\boldsymbol{\mathcal{Y}}_\theta &= \cos^2\theta\, \boldsymbol{\mathcal{Y}} + \sin\theta\cos\theta\,(\boldsymbol{\mathcal{X}} - \widetilde{\boldsymbol{\mathcal{Z}}}) - \sin^2\theta\, \boldsymbol{\mathcal{Y}}^T,\\
\widetilde{\boldsymbol{\mathcal{Z}}}_\theta &= \cos^2\theta\, \widetilde{\boldsymbol{\mathcal{Z}}} + \sin\theta\cos\theta\,(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \sin^2\theta\, \boldsymbol{\mathcal{X}}
\end{aligned} \tag{4.31}$$

with $\widetilde{\boldsymbol{\mathcal{Z}}} = \boldsymbol{\mathcal{Z}} - k^2\, v^2\, \mathbb{1}$.

3. *If $0 \le v < \widehat{v}$, then the solution $\mathcal{M}_v$ obtained from* (4.29) *has the following properties*
   (a) $\boldsymbol{\mathcal{M}}_v$ *is hermitian,*
   (b) $\frac{d\boldsymbol{\mathcal{M}}_v}{dv}$ *is negative definite,*
   (c) $\mathrm{tr}(\boldsymbol{\mathcal{M}}_v) \ge 0$*, and* $\langle w, \boldsymbol{\mathcal{M}}_v\, w\rangle \ge 0$ *for all real vectors w for all* $0 \le v \le \widehat{v}$*,*
   (d) $\boldsymbol{\mathcal{M}}_v$ *is positive definite.*

4. *The secular equation*

$$\det \boldsymbol{\mathcal{M}}_v = 0, \tag{4.32}$$

where $\boldsymbol{\mathcal{M}}_v$ is obtained from (4.29)*, has a unique admissible solution $0 \le v < \widehat{v}$. In other words there* **exists a unique Love wave propagating in the Cosserat medium**.

Proof. The reader may consult the paper by Fu and Mielke [10] since we have transformed the Love wave propagation problem in the framework so that the Fu and Mielke's results could be directly applied. Complete explanations in our notations may be found in [14], too. ■

From a computational perspective, the key advantage of the aforementioned result lies in the fact that, since $\boldsymbol{\mathcal{X}}_\theta$ and $\boldsymbol{\mathcal{Y}}_\theta$ depend on the wave speed v, the secular equation $\det \boldsymbol{\mathcal{M}}_v = 0$ can be formulated explicitly without requiring prior knowledge of the analytical expressions (as functions of the wave speed) for the eigenvalues satisfying (4.12) and the associated eigenvector $d^{(k)}$. This bypasses the primary challenge encountered in most generalised continuum models, except for specific cases where the task is relatively straightforward, such as classical isotropic linear elasticity [1] or the theory of materials with voids [34, 3, 37], provided more restrictive conditions are imposed on the constitutive coefficients.

5 Numerical implementation

In what follows we will do the calculations for dense polyurethane Foam@0.18mm with the constitutive parameters given in [24, 23]:

$$\begin{aligned}
\mu_{\mathrm{e}} = 104\,(\mathrm{MPa}), \qquad \mu_{\mathrm{c}} = 4.3331\,(\mathrm{MPa}), \qquad \mu_{\mathrm{e}}\, L_{\mathrm{c}}^2\, \alpha_1 = 79.9552\,(\mathrm{MPa}\cdot\mathrm{mm}^2),\\
\mu_{\mathrm{e}}\, L_{\mathrm{c}}^2\, \alpha_2 = 10.6496\,(\mathrm{MPa}\cdot\mathrm{mm}^2), \qquad \mu_{\mathrm{e}}\, L_{\mathrm{c}}^2\, \alpha_3 = -53.3035\,(\mathrm{MPa}\cdot\mathrm{mm}^2),\\
J = j\, \mu_{\mathrm{e}}\, \tau_{\mathrm{c}}^2 = 10\,(\mathrm{mm}^2), \qquad \rho_0 = 340\cdot 10^{-6}\,(\mathrm{g/mm}^3),
\end{aligned} \tag{5.1}$$

and we take the wave number $k = 0.5\,(\mathrm{mm}^{-1})$.

5.1 Numerical implementation for Love waves

Imposing that the roots of the characteristic equation (4.12) not to be real, one can determine the speed v by numerical calculation. We start form 0 with a small step size 10^{-12} and we find the first value of v for which (4.12) admits a real solution. This is the numerical approximation of the limiting speed. For the considered material the value of the limiting speed is approximated to be $\widehat{v} \in (6.77866, 6.77867)$ m/s.

To find the solution of the secular equation (4.32) we compute $\boldsymbol{\mathcal{M}}_v$ using (4.30) by interpolating over points in $[0, \widehat{v})$. Since we have remarked that the solutions are in a small neighbourhood of the limiting speed, we have considered for interpolation of 100 intermediate points which are in a very small neighbourhood of the limiting speed. In this way we find the approximate solution of the secular equation $v_0 = 1.52374204511`$ m/s.

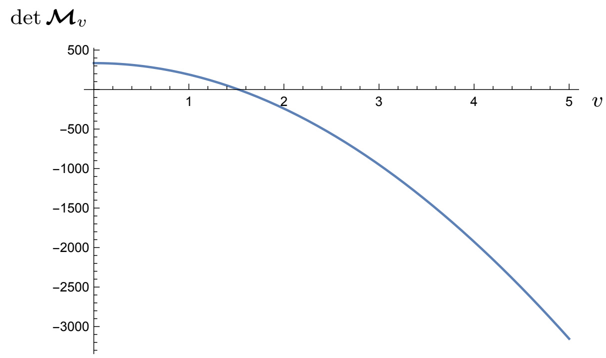


Figure 2: A plot $(v, \det \boldsymbol{\mathcal{M}}_v)$ obtained by evaluation for the given equidistant values for speed for dense polyurethane Foam @0.18mm. The function $v \mapsto \det \boldsymbol{\mathcal{M}}_v$ is a decreasing function of the wave speed v.

Using the main theorem 4.4, we find the corresponding matrix $\boldsymbol{\mathcal{M}}_{v_0}$ to be

$$\boldsymbol{\mathcal{M}}_{v_0} = \begin{pmatrix} 78.92996132702877 & 1.5713435539151161i & 1.971330465220131 \\ -1.5713435539151161i & 8.968058122320638 & -9.11569303802329i \\ 1.971330465220131 & 9.1156930380232i & 9.267537853294720 \end{pmatrix}. \tag{5.2}$$

The matrix $\boldsymbol{\mathcal{E}}_{v_0} = \boldsymbol{\mathcal{X}}^{-1}(\boldsymbol{\mathcal{M}}_{v_0} + \mathrm{i}\, \boldsymbol{\mathcal{Y}}^T)$ is then numerically approximated by

$$\boldsymbol{\mathcal{E}}_{v_0} = \begin{pmatrix} 78.92996132702877 & 1.5713435539151161i & 1.971330465220131 \\ -1.5713435539151161i & 8.968058122320638 & -9.11569303802329i \\ 1.9713304652201319 & 9.11569303802329i & 9.267537853294720 \end{pmatrix}. \tag{5.3}$$

We determine $y(x_2)$ in the form given by (4.23), i.e., a solution of the initial value problem (4.24), after we find its approximate initial value

$$y(0) = \begin{pmatrix} y_1 \\ -213.52544299420iy_1 \\ -210.23962949514y_1 \end{pmatrix}, \qquad y_1 \in \mathbb{C}. \tag{5.4}$$

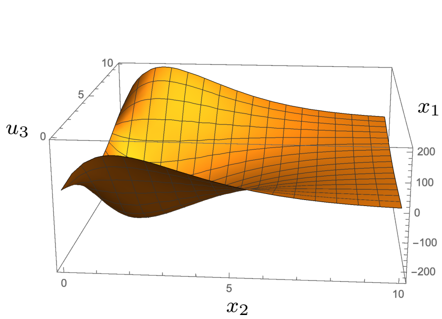
After a substitution $y(x_2) := \widehat{\mathbb{1}}^{1/2}\, z(x_2)$ which lead us to $z(x_2)$ and then to the solution describing the Love waves propagation

$$\mathcal{U}(x_1, x_2, t) = \begin{pmatrix} u_3(x_1, x_2, t) \\ \vartheta_1(x_1, x_2, t) \\ \vartheta_2(x_1, x_2, t) \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} \mathrm{i}\, z_3(x_2) \\ z_1(x_2) \\ z_2(x_2) \end{pmatrix} e^{\mathrm{i}\, k\, (x_1 - vt)}\right],$$

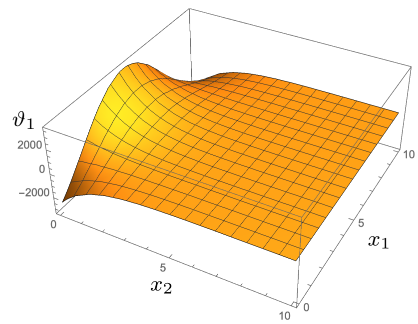
of the problem given by (3.8) and (3.9), see Figure 3.

5.2 Numerical implementation for Rayleigh waves

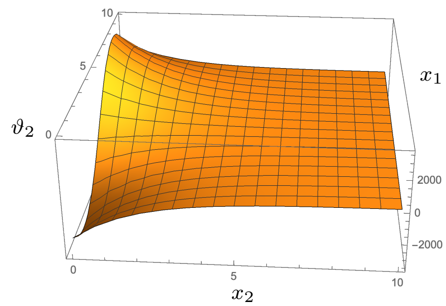
For comparison, we approximate the solution of the Rayleigh waves. The similar approach for the case of Rayleigh waves in isotropic elastic Cosserat materials can be found in [14].



(a) Plot of the u_3-component of the displacement.



(b) Plot of ϑ_1-component of the micro-rotation vector.



(c) Plot of ϑ_2-component of the micro-rotation vector.

Figure 3: The plot of the solution at time $t = 1$ and for the choice $y_1 = 1$.

We find the the limiting speed is such that $\widehat{v} \in [6.78866, 6.78868]$ m/s, while the approximate solution of the secular equation is $v_0 = 6.785867864122372$ m/s.

We give the plot of the approximate solution $\mathcal{U} = (u_1, u_2, \vartheta_3)$ for Rayleigh waves in Figure 4.

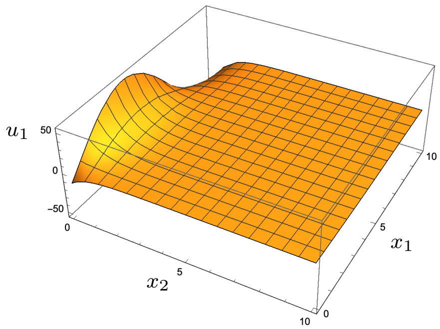
In the end of this subsection we propose different values of J and we calculate the speed of wave that gave the solution for the secular equation (4.27) for Love waves and also for Rayleigh wave, see Tables 1 and 2.

	J=0.1	J=1	J=10
speed of Love waves	14.0123177251145498	4.79994344526609	1.523742045114
speed of Rayleigh waves	17.591996478068255	17.31368477726568	6.78586786412209395

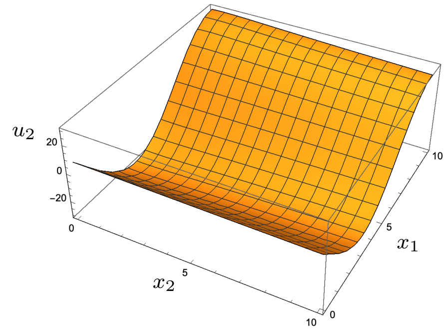
Table 1: The values of solution of the secular equation for different J when $k = 0.5$ in the case of Love and Rayleigh waves.

	k=0.5	k=5
speed of Love waves	4.79994344526609	16.352013950835141
speed of Rayleigh waves	17.31368477726568	16.37076505452122395

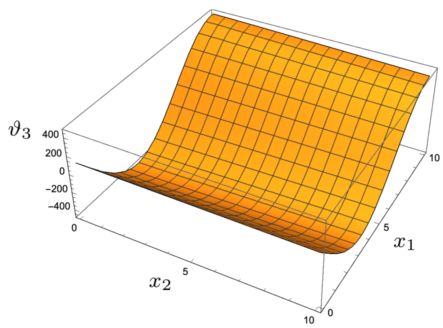
Table 2: The values of solution of the secular equation for different k when $J = 1$ for Love and Rayleigh waves.



(a) Plot of the u_1-component of the displacement.



(b) Plot of u_2-component of the micro-rotation vector.



(c) Plot of ϑ_3-component of the micro-rotation vector.

Figure 4: The plot of the solution $\mathcal{U} = (u_1, u_2, \vartheta_3)$ in the case of Rayleigh waves at time $t = 1$ and for the choice $\varsigma = \mathrm{i}$.

6 From linear Cosserat theory to classical linear elasticity: a consistency check and comparison of the results

In this section we rediscover the results from classical linear elasticity as a limit case of the results obtained in the linear Cosserat theory. First, we observe that if $\mu_c \to 0$, then the total energy (2.1) is decoupled into the energy due to the macro-displacement u

$$W^{\rm macro}(u) := \rho\,\|u_{,t}\|^2 + \mu_{\rm e}\,\|{\rm dev}_3\,{\rm sym}\,{\rm D}u\|^2 + \frac{2\,\mu_{\rm e} + 3\,\lambda_{\rm e}}{6}\,[{\rm tr}\,({\rm D}u)]^2\,, \tag{6.1}$$

and the energy due to the micro-rotation ϑ

$$W^{\rm micro}(\vartheta) :=\ \rho\,j\,\mu_{\rm e}\,\tau_{\rm c}^2\,\|\vartheta_{,t}\|^2 + \frac{\mu_{\rm e} L_{\rm c}^2}{2}\left[a_1\|{\rm dev\,sym\,D}\vartheta)\|^2 + a_2\|{\rm skew\,D}\vartheta\|^2 + \frac{4\,a_3}{3}\,[{\rm tr}({\rm D}\vartheta)]^2\right]. \tag{6.2}$$

Then, we have that the total energy remains finite for $L_{\rm c} \to \infty$, if $\|{\rm D}\vartheta\|^2 = 0$ which further implies that $\vartheta =$ constant in Ω. This situation corresponds to a time dependent rigid (macroscopic) movement of the entire body. In addition, under Dirichlet homogeneous boundary conditions on ϑ we find that ϑ vanishes in the entire body at any time.

Moreover, assuming also that $\tau_{\rm c} \to 0$, too, the inertia term due to the micro-rotation vanishes.

Therefore, all these limit cases together lead us back in the framework of classical linear elasticity, i.e., the elastic energy density is

$$W_{\rm classic}({\rm D}u) = \mu_{\rm e}\,\|{\rm dev}_3\,{\rm sym}\,{\rm D}u\|^2 + \frac{2\,\mu_{\rm e} + 3\,\lambda_{\rm e}}{6}\,[{\rm tr}\,({\rm D}u)]^2\,, \tag{6.3}$$

while the equations of motion are the Lamé equations

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} = \underbrace{\mu_{\rm e}\sum_{l=1}^{3}\frac{\partial^2 u_i}{\partial x_l^2} + (\mu_{\rm e}+\lambda_{\rm e})\sum_{l=1}^{3}\frac{\partial^2 u_l}{\partial x_l\partial x_i}}_{={\rm Div}\boldsymbol{\sigma}_{\rm classic}}, \tag{6.4}$$

and the decoupled equations of micro-rotations

$$\rho\,j\,\mu_{\rm e}\,\tau_{\rm c}^2\,\frac{\partial^2 \vartheta_i}{\partial t^2} = \underbrace{\mu_{\rm e}L_{\rm c}^2\left[(\alpha_1+\alpha_2)\sum_{l=1}^{3}\frac{\partial^2\vartheta_i}{\partial x_l^2} + (\alpha_1-\alpha_2+\alpha_3)\sum_{l=1}^{3}\frac{\partial^2\vartheta_l}{\partial x_l\partial x_i}\right]}_{={\rm Div}\mathbf{m}},$$

with $i=1,2,3$ and

$$\begin{aligned}\boldsymbol{\sigma}_{\rm classic} &:= 2\,\mu_{\rm e}\,{\rm sym}\,{\rm D}u\, + \lambda_{\rm e}\,{\rm tr}({\rm sym}\,{\rm D}u)\,\mathbb{1},\\ \boldsymbol{m}_{\rm microrotation} &:= \mu_{\rm e}\,L_{\rm c}^2\,\left[2\,\alpha_1\,{\rm sym}\,{\rm D}\vartheta + 2\,\alpha_2\,{\rm skew}\,{\rm D}\vartheta + \alpha_3\,{\rm tr}({\rm D}\vartheta)\,\mathbb{1}\right].\end{aligned} \tag{6.5}$$

Letting $\mu_{\rm c}\to 0$, the matrices involved in the Riccati equation become

$$\begin{aligned}\boldsymbol{\mathcal{X}} &:= \begin{pmatrix}\frac{k^2\mu_e}{\rho} & 0 & 0\\ 0 & \frac{k^2L_c^2(\alpha_1+\alpha_2)}{\rho\tau_c^2 j} & 0\\ 0 & 0 & \frac{k^2L_c^2(2\alpha_1+\alpha_3)}{\rho\tau_c^2 j}\end{pmatrix}, \qquad \boldsymbol{\mathcal{Y}} := \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \frac{\alpha_3 L_c^2 k}{\rho\tau_c^2 j}\\ 0 & \frac{kL_c^2(\alpha_1-\alpha_2)}{\rho\tau_c^2 j} & 0\end{pmatrix},\\ \widetilde{\boldsymbol{\mathcal{Z}}} &:= \begin{pmatrix}\frac{k^2\mu_e}{\rho}-k^2v^2 & 0 & 0\\ 0 & \frac{k^2L_c^2(2\alpha_1+\alpha_3)}{\rho\tau_c^2 j}-k^2v^2 & 0\\ 0 & 0 & \frac{k^2L_c^2(\alpha_1+\alpha_2)}{\rho\tau_c^2 j}-k^2v^2\end{pmatrix}.\end{aligned} \tag{6.6}$$

Thus, we have

$$\boldsymbol{\mathcal{X}}_\theta = \begin{pmatrix}\frac{k^2\left(\mu_e-\rho v^2\sin^2(\theta)\right)}{\rho} & 0^T\\ 0 & \boldsymbol{\mathcal{A}}_\theta\end{pmatrix}, \qquad \boldsymbol{\mathcal{Y}}_\theta = \begin{pmatrix}k^2v^2\sin(\theta)\cos(\theta) & 0^T\\ 0 & \boldsymbol{\mathcal{B}}_\theta\end{pmatrix}, \tag{6.7}$$

where

$$\begin{aligned}\boldsymbol{\mathcal{A}}_\theta &= \begin{pmatrix}k^2\left(\alpha_2c^2\cos^2(\theta)-\frac{1}{2}\alpha_1c^2(\cos(2\theta)-3)+\sin^2(\theta)\left(\alpha_3c^2-v^2\right)\right) & (\alpha_1-\alpha_2+\alpha_3)\left(-c^2\right)k\sin(\theta)\cos(\theta)\\ (\alpha_1-\alpha_2+\alpha_3)\left(-c^2\right)k\sin(\theta)\cos(\theta) & \frac{1}{2}k^2\left(c^2\left(\alpha_1(\cos(2\theta)+3)+2\left(\alpha_2\sin^2(\theta)+\alpha_3\cos^2(\theta)\right)\right)-2v^2\sin^2(\theta)\right)\end{pmatrix}\\ \boldsymbol{\mathcal{B}}_\theta &= \begin{pmatrix}k^2\sin(\theta)\cos(\theta)\left(v^2-(\alpha_1-\alpha_2+\alpha_3)\,c^2\right) & c^2k\left((\alpha_2-\alpha_1)\sin^2(\theta)+\alpha_3\cos^2(\theta)\right)\\ c^2k\left((\alpha_1-\alpha_2)\cos^2(\theta)-\alpha_3\sin^2(\theta)\right) & k^2\sin(\theta)\cos(\theta)\left((\alpha_1-\alpha_2+\alpha_3)\,c^2+v^2\right)\end{pmatrix},\end{aligned} \tag{6.8}$$

with the notation $c=\frac{L_c^2}{\rho\tau_c^2 j}$.

If we impose the boundary condition

$$\boldsymbol{\sigma}_{\rm classic}.\,n = 0, \qquad \text{for}\quad x_2=0, \tag{6.9}$$

and the following decay condition

$$\lim_{x_2\to\infty}\{u_3,\sigma_{13}^{\rm classic},\sigma_{23}^{\rm classic}\}(x_1,x_2,t)=0 \qquad \forall\,x_1\in\mathbb{R},\quad \forall\,t\in[0,\infty), \tag{6.10}$$

by following the usual path from classical elasticity, i.e., by looking for a solution

$$\mathcal{U}_{\rm classic}(x_1,x_2,t) = \begin{pmatrix}u_1(x_1,x_2,t)\\ u_2(x_1,x_2,t)\\ u_3(x_1,x_2,t)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ d_3^{\rm classic}(x_1,x_2,t)\end{pmatrix} = {\rm Re}\left[\begin{pmatrix}0\\ 0\\ d_3^{\rm classic}\end{pmatrix}\,e^{{\rm i}\,r\,k\,x_2}\,e^{{\rm i}\,k\,(x_1-vt)}\right], \quad {\rm Im}\,r>0, \tag{6.11}$$

of (6.4), it is easy to see [1] that

$$r=\sqrt{1-\left(\frac{v}{c_T}\right)^2}, \qquad \text{where}\quad c_T=\sqrt{\frac{\mu_{\rm e}}{\rho}}. \tag{6.12}$$

Moreover, since from the free surface boundary conditions on $x_2 = 0$ we have

$$\frac{\partial u_3}{\partial x_2}\Big|_{x_2=0} = 0, \tag{6.13}$$

we find that either $d_3^{\text{classic}} = 0$ or $r = 0$, neither case representing a surface wave, see [1, page 219]. In conclusion, using the classical linear theory of isotropic elastic materials it is not possible to analytically find that surface waves with displacements perpendicular to the plane of propagation araise [1]. However, experimental data shows that these kind of waves may occur along a free surface. A compromise has been introduced in classical elasticity, see [1, page 219], by considering that the half space is covered by a thin layer of a different material. For this new problem it is proved that Love-type waves orise [1, page 219]. However, there is no need to consider layers in order to explain the existence of Love waves from experiments, since the presence of the Love waves could be explained theoretical by the presence of micro-rotations, in the sense that even in the uncoupled case ($\mu_c = 0$) the displacement u_3 may not vanish for a speed of propagation produced by the microstructure. We will explain this in the rest of this section.

We may argue that c_T is not an admissible speed of propagation in the sense of our approach, i.e., by showing that it is bigger than the limiting speed. Now, by the limiting speed we understand a speed $\widehat{v} > 0$, such that for all wave speeds satisfying $0 \le v < \widehat{v}$ (subsonic speeds) the roots of the characteristic equation (4.12) corresponding to the matrices given by (6.6) are not real and vice versa, i.e., if the roots of the characteristic equation (6.6) are not real then they correspond to wave speeds v satisfying $0 \le v < \widehat{v}$.

Indeed, similar arguments as those used for the proof of Proposition 2.1 and Proposition 4.3 lead us to

Proposition 6.1. *Let $\xi \in \mathbb{R}^3$, $\xi \neq 0$ be any direction of the form $\xi = (\xi_1, \xi_2, 0)^T$. The necessary and sufficient conditions for existence of a non trivial solution of the system of partial differential equations* (6.4)*,* (6.5) *of the form given by* (2.7) *are*

$$2\alpha_1 + \alpha_3 > 0, \qquad \mu_e > 0, \qquad \alpha_1 + \alpha_2 > 0. \tag{6.14}$$

Let us assume that the constitutive coefficients satisfy the conditions (2.11). Then, there exists a limiting speed $\widehat{v} > 0$. Furthermore, if one root r_v of the characteristic equation (4.12) coresponding to the matrices given by (6.6) is real then it corresponds to a speed $v \ge \widehat{v}$ (non-admissible). Moreover, for all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $k > 0$ and $0 \le v < \widehat{v}$, the tensor $\widetilde{\boldsymbol{\mathcal{Z}}}_\theta := \sin^2\theta\, \boldsymbol{\mathcal{X}} + \sin\theta\cos\theta(\boldsymbol{\mathcal{Y}} + \boldsymbol{\mathcal{Y}}^T) + \cos^2\theta\, \boldsymbol{\mathcal{Z}} - k^2 v^2 \cos^2\theta\, \mathbb{1}$ is positive definite. Since $\boldsymbol{\mathcal{X}}_\theta(\theta + \frac{\pi}{2}) = \widetilde{\boldsymbol{\mathcal{Z}}}_\theta$, we find that for all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $k > 0$ and $0 \le v < \widehat{v}$, the tensor $\boldsymbol{\mathcal{X}}_\theta$ must be positive definite. Therefore, we must have

$$\frac{k^2\left(\mu_e - \rho v^2 \sin^2(\theta)\right)}{\rho} > 0 \qquad \forall\, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \forall\, k > 0, \quad \forall\, 0 \le v < \widehat{v}, \tag{6.15}$$

which implies that

$$v < c_T = \sqrt{\frac{\mu_e}{\rho}} \qquad \forall\, 0 \le v < \widehat{v}, \tag{6.16}$$

i.e., it implies that $\widehat{v} \le c_T$ and that an admissible speed of propagation may never reach the value c_T.

Consequently, from (4.30) we deduce that the matrix $\boldsymbol{\mathcal{M}}$ giving the secular equation $\det \boldsymbol{\mathcal{M}} = 0$ is

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \Big[\displaystyle\int_0^\pi \Big(\frac{k^2\left(\mu_e - \rho v^2 \sin^2(\theta)\right)}{\rho}\Big)^{-1} d\theta\Big]^{-1} \Big[\pi\mathbb{1} - \mathrm{i}\displaystyle\int_0^\pi \Big(\frac{k^2\left(\mu_e - \rho v^2 \sin^2(\theta)\right)}{\rho}\Big)^{-1} k^2 v^2 \sin(\theta)\cos(\theta)\, d\theta\Big] & 0^T \\ 0 & \boldsymbol{\mathcal{N}} \end{pmatrix}, \tag{6.17}$$

where

$$\boldsymbol{\mathcal{N}} = \Big(\int_0^\pi \boldsymbol{\mathcal{A}}_\theta^{-1}\, d\theta\Big)^{-1}\Big(\pi\mathbb{1} - \mathrm{i}\int_0^\pi \boldsymbol{\mathcal{A}}_\theta^{-1}\boldsymbol{\mathcal{B}}_\theta^T\, d\theta\Big), \tag{6.18}$$

We have

$$\begin{aligned} &\Big[\int_0^\pi \Big(\frac{k^2\left(\mu_e - \rho v^2 \sin^2(\theta)\right)}{\rho}\Big)^{-1} d\theta\Big]^{-1} \Big[\pi\mathbb{1} - \mathrm{i}\int_0^\pi \Big(\frac{k^2\left(\mu_e - \rho v^2 \sin^2(\theta)\right)}{\rho}\Big)^{-1} k^2 v^2 \sin(\theta)\cos(\theta)\, d\theta\Big] \\ &\qquad = \frac{2ik^2\sqrt{\mu_e}\sqrt{\rho v^2 - \mu_e}}{\rho\,(2\mu_e - \rho v^2 + 2\pi)} \neq 0. \end{aligned} \tag{6.19}$$

Therefore, since we already know that for $\mu_c \to 0$ (from the same arguments as for the case $\mu_c > 0$) there is a unique solution of the secular equation $\det \boldsymbol{\mathcal{N}}$, using (6.19) we have that the secular equation reduces to

$$\det \boldsymbol{\mathcal{N}} = 0, \tag{6.20}$$

where $\boldsymbol{\mathcal{N}}$ is given by (6.18) and that there is a unique solution of this equation which satisfies $0 \leq v < \widehat{v}$. In particular the solution v is not c_T, which implies that u_3 does not vanish, i.e., that we have a theoretical meaning of the existence of the Love waves in experiments, without considering additional material layers as it is done before [1]. This confirms and proofs rigorously the remark made in [17, 21, 22] about the existence of a qualitatively new wave mode for the Cosserat elastic model. The meaning is that the non-zero macro-displacement u_3 is produced by a speed of propagation which is due to the given microstructure.

The exact form of $\boldsymbol{\mathcal{N}}$ is

$$\boldsymbol{\mathcal{N}} = \begin{pmatrix} \dfrac{c v^2}{c^2\sqrt{\frac{2\alpha_1+\alpha_3}{(2\alpha_1+\alpha_3)c^2-v^2}}-\sqrt{c^2-\frac{v^2}{\alpha_1+\alpha_2}}} & -\dfrac{i c^2\sqrt{\frac{2\alpha_1+\alpha_3}{(2\alpha_1+\alpha_3)c^2-v^2}}\,2\alpha_1\left(c^2-\sqrt{\frac{\left(c^2-\frac{v^2}{\alpha_1+\alpha_2}\right)\left((2\alpha_1+\alpha_3)c^2-v^2\right)}{2\alpha_1+\alpha_3}}-v^2\right)}{c^2\sqrt{\frac{2\alpha_1+\alpha_3}{(2\alpha_1+\alpha_3)c^2-v^2}}-\sqrt{c^2-\frac{v^2}{\alpha_1+\alpha_2}}} \\ \dfrac{i c^2\left(-v^2\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)c^2-v^2}}-2\alpha_1\left(\sqrt{c^2-\frac{v^2}{2\alpha_1+\alpha_3}}-c^2\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)c^2-v^2}}\right)\right)}{c^2\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)c^2-v^2}}-\sqrt{c^2-\frac{v^2}{2\alpha_1+\alpha_3}}} & \dfrac{c v^2}{c^2\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)c^2-v^2}}-\sqrt{c^2-\frac{v^2}{2\alpha_1+\alpha_3}}} \end{pmatrix}.. \tag{6.21}$$

Let us remark that since for all $v \in [0, \widehat{v})$ the matrix $\boldsymbol{\mathcal{A}}_\theta$ is also positive definite $\forall \theta \in [0, \pi]$, then $\int_0^\pi \boldsymbol{\mathcal{A}}_\theta^{-1}\, d\theta$ is positive definite. Since

$$\int_0^\pi \boldsymbol{\mathcal{A}}_\theta^{-1}\, d\theta = \begin{pmatrix} \dfrac{\pi\left(c^2\sqrt{\frac{2\alpha_1+\alpha_3}{(2\alpha_1+\alpha_3)c^2-v^2}}-\sqrt{c^2-\frac{v^2}{\alpha_1+\alpha_2}}\right)}{c v^2} & 0 \\ 0 & \dfrac{\pi c^2\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)c^2-v^2}}-\pi\sqrt{c^2-\frac{v^2}{2\alpha_1+\alpha_3}}}{c v^2} \end{pmatrix} \in \mathbb{R}^{2\times 2} \tag{6.22}$$

it follows that for all $v \in [0, \widehat{v})$ we must have

$$(2\alpha_1+\alpha_3)\, c^2 - v^2 > 0, \ (\alpha_1+\alpha_2)\, c^2 - v^2 > 0 \text{ and } c^2\sqrt{\frac{2\alpha_1+\alpha_3}{(2\alpha_1+\alpha_3)\, c^2 - v^2}}\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)\, c^2 - v^2}} > 1. \tag{6.23}$$

The third inequality is redundant, while if $(6.23)_{1,2}$ are satisfied, then $\boldsymbol{\mathcal{N}}$ is an hermitian matrix, while the secular equation is given by $f_c(v) = 0$, with $f_c : [0, \widehat{v}) \to \mathbb{R}$,

$$\begin{aligned} f_c(v) := & \frac{1}{\left(c^2\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)c^2-v^2}}-\sqrt{c^2-\frac{v^2}{2\alpha_1+\alpha_3}}\right)\left(\sqrt{c^2-\frac{v^2}{\alpha_1+\alpha_2}}-c^2\sqrt{\frac{2\alpha_1+\alpha_3}{(2\alpha_1+\alpha_3)c^2-v^2}}\right)} \\ & \times \Big\{ c^2 v^4 \left(c^2\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)\, c^2 - v^2}}\sqrt{\frac{2\alpha_1+\alpha_3}{(2\alpha_1+\alpha_3)\, c^2 - v^2}} - 1 \right) \\ & \quad + 2\alpha_1 c^4 (v^2 \left(\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)\, c^2 - v^2}} \left(\sqrt{\frac{\left((\alpha_1+\alpha_2)\, c^2 - v^2\right)\left((2\alpha_1+\alpha_3)\, c^2 - v^2\right)}{(\alpha_1+\alpha_2)\,(2\alpha_1+\alpha_3)}} - 2c^2 \right) + \sqrt{c^2 - \frac{v^2}{2\alpha_1+\alpha_3}} \right) \\ & \quad - 2\alpha_1 \left(c^2\sqrt{\frac{\alpha_1+\alpha_2}{(\alpha_1+\alpha_2)\, c^2 - v^2}} - \sqrt{c^2 - \frac{v^2}{2\alpha_1+\alpha_3}} \right) \left(\sqrt{\frac{\left((\alpha_1+\alpha_2)\, c^2 - v^2\right)\left(c^2 - \frac{v^2}{2\alpha_1+\alpha_3}\right)}{\alpha_1+\alpha_2}} - c^2 \right)) \\ & \quad \times \sqrt{\frac{2\alpha_1+\alpha_3}{(2\alpha_1+\alpha_3)\, c^2 - v^2}} \Big\}. \end{aligned} \tag{6.24}$$

Let us remark that for the assumptions from the beginning of this section, i.e., $L_c \to \infty$ and $\tau_c \to 0$ it is natural to consider the case $c \to \infty$, the expectation being that is this situation we must lead to the same result as in the classical elasticity. Indeed, we already know that there is no admissible speed due to the macrostructure, but we also have that assuming (6.23), we have that $\lim_{c\to\infty} f_c(v) = \infty$, $\forall v \in [0, \widehat{v})$. This means that in this case there does not exist a solution of the secular equation due to the microstructure. Therefore, for Love waves propagation $\mu_c \to 0$, $L_c \to \infty$ and $\tau_c \to 0$ means the framework of classical elasticity.

7 Conclusion

This paper, together with [12], makes possible a complete comparison between the propagation of two principal types of surface waves, Love waves and Rayleigh waves, in Cosserat homogeneous isotropic materials. From experiments we know that, in general, they exhibit fundamentally distinct modes of motion. Love waves are characterised by purely horizontal, transverse displacements perpendicular to the direction of wave propagation, with no vertical component. In contrast, Rayleigh waves induce retrograde elliptical particle motion in the vertical plane, combining both vertical and longitudinal displacements along the direction of propagation. In the standard half-space model used to describe these waveforms, the boundary is assumed to be traction-free, and all non-zero displacement and stress components are required to decay asymptotically to zero with increasing depth. The method and the algorithms considered in [12] and the present paper are easy to be implemented and mathematically fully justified, due to the previous results given by Fu and Mielke [11, 29].

It is known that the experimental data shows that Love waves exists. It is also known that in the framework of classical linear theory of isotropic elastic materials it is not possible to find analytically Love waves-type solutions [1]. To avoid this fact, a compromise has been introduced in classical elasticity, see [1, page 219], by considering that the half space is covered by a layer of a different material. In this paper, we give a complete mathematical study which shows that the Cosserat theory is able to model the Love waves propagation without any artificial introduction of surface layers.

A complete mathematical study of the propagation of Love waves in Cosserat elastic materials which includes a proof of the existence and uniqueness of an admissible solution of the secular equation was missing until our study. The present study confirms the effectiveness of Fu and Mielke's general method in establishing the existence and uniqueness of a subsonic solution that describes the propagation of Love waves in an isotropic half-space (without additional material surface layers), based on the linear theory of isotropic elastic Cosserat materials. The given formulas and the given algorithm admit numerical implementation and we use them for the study of Love waves propagation in dense polyurethane Foam@0.18mm [24, 23]. We remark that once in the measurement of Love waves and Rayleigh waves propagation on the Earth surface the Love waves propagate faster than the Rayleigh waves, for our material modelled in the framework of the Cosserat linear theory we obtain the opposite. For other materials modelled by the Cosserat linear theory, numerical simulations show that there is no general rule about which of those type of seismic waves travel faster, this property depending on the material parameters and needs further research. We have also seen that for Love waves propagation the case $\mu_c \to 0$, $L_c \to \infty$ and $\tau_c \to 0$ means the framework of classical elasticity, while, in general, for $\mu_c \to 0$ the presence of the Love waves at macroscopic level could be an effect of the microscopic level.

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A Notation

In the following, we recall some useful notations for the present work. For $a, b \in \mathbb{R}^{3\times 3}$ we let $\langle a, b\rangle_{\mathbb{R}^3}$ denote the scalar product on $\mathbb{R}^3$ with associated vector norm $\|a\|^2 = \langle a, a\rangle$. We denote by $\mathbb{R}^{3\times 3}$ the set of real 3×3 second order tensors, written with capital letters. Matrices will be denoted by bold symbols, e.g. $\mathbf{X} \in \mathbb{R}^{3\times 3}$, while X_{ij} will denote its component. The standard Euclidean product on $\mathbb{R}^{3\times 3}$ is given by $\langle \mathbf{X}, \mathbf{Y}\rangle_{\mathbb{R}^{3\times 3}} = \mathrm{tr}(\mathbf{X}\,\mathbf{Y}^T)$, and thus, the Frobenious tensor norm is $\|\mathbf{X}\|^2 = \langle \mathbf{X}, \mathbf{X}\rangle_{\mathbb{R}^{3\times 3}}$. In the following we omit the index $\mathbb{R}^3, \mathbb{R}^{3\times 3}$. The identity tensor on $\mathbb{R}^{3\times 3}$ will be denoted by $\mathbb{1}$, so that $\mathrm{tr}(\mathbf{X}) = \langle \mathbf{X}, \mathbb{1}\rangle$. We let Sym denote the set of symmetric tensors. We adopt the usual abbreviations of Lie-algebra theory, i.e., $\mathfrak{so}(3) := \{\mathbf{A} \in \mathbb{R}^{3\times 3} | \mathbf{A}^T = -\mathbf{A}\}$ is the Lie-algebra of skew-symmetric tensors and $\mathfrak{sl}(3) := \{\mathbf{X} \in \mathbb{R}^{3\times 3} | \mathrm{tr}(\mathbf{X}) = 0\}$ is the Lie-algebra of traceless tensors. For all $\mathbf{X} \in \mathbb{R}^{3\times 3}$ we set $\mathrm{sym}\,\mathbf{X} = \frac{1}{2}(\mathbf{X}^T + \mathbf{X}) \in \mathrm{Sym}$, $\mathrm{skew}\mathbf{X} = \frac{1}{2}(\mathbf{X} - \mathbf{X}^T) \in \mathfrak{so}(3)$ and the deviatoric (trace-free) part $\mathrm{dev}\,\mathbf{X} = \mathbf{X} - \frac{1}{3}\mathrm{tr}(\mathbf{X}) \in \mathfrak{sl}(3)$ and we have the orthogonal Cartan-decomposition of the Lie-algebra $\mathfrak{gl}(3) = \{\mathfrak{sl}(3) \cap \mathrm{Sym}(3)\} \oplus \mathfrak{so}(3) \oplus \mathbb{R}\cdot\mathbb{1},\ \mathbf{X} = \mathrm{dev}\,\mathrm{sym}\,\mathbf{X} + \mathrm{skew}\,\mathbf{X} + \frac{1}{3}\mathrm{tr}(\mathbf{X})\,\mathbb{1}$. We use the canonical identification of $\mathbb{R}^3$ with $\mathfrak{so}(3)$, and, for

$$\mathbf{A} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in \mathfrak{so}(3) \tag{A.25}$$

we consider the operators $\mathrm{axl} : \mathfrak{so}(3) \to \mathbb{R}^3$ and $\mathbf{Anti} : \mathbb{R}^3 \to \mathfrak{so}(3)$ through

$$\begin{aligned} &\mathrm{axl}\,(\mathbf{A}) : = (a_1, a_2, a_3)^T\,, \qquad \mathbf{A}.\,v = (\mathrm{axl}\,\mathbf{A}) \times v, \qquad (\mathbf{Anti}(v))_{ij} = -\epsilon_{ijk}\,v_k, \qquad \forall\, v \in \mathbb{R}^3, \\ &(\mathrm{axl}\,\mathbf{A})_k = -\frac{1}{2}\,\epsilon_{ijk}\mathbf{A}_{ij} = \frac{1}{2}\,\epsilon_{kij}A_{ji}\,, \quad A_{ij} = -\epsilon_{ijk}\,(\mathrm{axl}\,\mathbf{A})_k =: \mathbf{Anti}(\mathrm{axl}\,\mathbf{A})_{ij}, \end{aligned} \tag{A.26}$$

where ϵ_{ijk} is the totally antisymmetric third order permutation tensor.

For a regular enough function $f(t, x_1, x_2, x_3)$, $f_{,t}$ denotes the derivative with respect to the time t, while $\frac{\partial f}{\partial x_i}$ denotes the i-component of the gradient ∇f. For vector fields $u = (u_1, u_2, u_3)^T$ with $u_i \in \mathrm{H}^1(\Omega) = \{u_i \in \mathrm{L}^2(\Omega)\,|\,\nabla\,u_i \in \mathrm{L}^2(\Omega)\}$, $i = 1, 2, 3$, we define $\mathrm{D}u := (\nabla\,u_1\,|\,\nabla\,u_2\,|\,\nabla\,u_3)^T$. The corresponding Sobolev-space will be also denoted by $\mathrm{H}^1(\Omega)$. In addition, for a tensor field $\mathbf{P}$ with rows in $\mathrm{H}(\mathrm{div}\,;\Omega)$, i.e., $\mathbf{P} = \big(\mathbf{P}^T.e_1\,|\,\mathbf{P}^T.e_2\,|\,\mathbf{P}^T\,e_3\big)^T$ with $(\mathbf{P}^T.e_i)^T \in \mathrm{H}(\mathrm{div}\,;\Omega) := \{v \in \mathrm{L}^2(\Omega)\,|\,\mathrm{div}\,v \in \mathrm{L}^2(\Omega)\}$, $i = 1, 2, 3$, we define $\mathrm{Div}\,\mathbf{P} := \big(\mathrm{div}\,(\mathbf{P}^T.e_1)^T\,|\,\mathrm{div}\,(\mathbf{P}^T.e_2)^T\,|\,\mathrm{div}\,(\mathbf{P}^T\,e_3)^T\big)^T$ while for tensor fields $\mathbf{P}$ with rows in $\mathrm{H}(\mathrm{curl}\,;\Omega)$, i.e., $\mathbf{P} = \big(\mathbf{P}^T.e_1\,|\,\mathbf{P}^T.e_2\,|\,\mathbf{P}^T\,e_3\big)^T$ with $(\mathbf{P}^T.e_i)^T \in \mathrm{H}(\mathrm{curl}\,;\Omega) := \{v \in \mathrm{L}^2(\Omega)\,|\,\mathrm{curl}\,v \in \mathrm{L}^2(\Omega)\}$, $i = 1, 2, 3$, we define $\mathrm{Curl}\,\mathbf{P} := \big(\mathrm{curl}\,(\mathbf{P}^T.e_1)^T\,|\,\mathrm{curl}\,(\mathbf{P}^T.e_2)^T\,|\,\mathrm{curl}\,(\mathbf{P}^T\,e_3)^T\big)^T$.